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Interactive Calculus Learning: merging Cognitive Roots, Documentational Genesis, and GeoGebra

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ABSTRACT

Background: This article aims to contribute to Calculus teaching regarding the selection of teaching resources that take learning difficulties into consideration. **Objectives:** We advocate choosing appropriate resources that reflect the results of theories and the advantages of using technology. **Design:** Theoretical constructs proposed by David Tall (1941-2024), such as cognitive root and local straightness, as well as documentational genesis, introduced by Gueudet and Trouche, are used to support our approach to the concepts of derivative and differential equation. **Data Collection and Analysis**: The study focuses on theoretical frameworks and practical examples of teaching resources developed. **Results:** Examples such as the blancmange function and the use of GeoGebra are discussed as tools that foster the understanding of complex concepts through interactive and dynamic visualizations. **Conclusions:** This study emphasizes the importance of combining theory and practice in the use of technologies to facilitate meaningful Calculus learning, which highlights the relevance of these approaches to the field of Mathematics Education.

Keywords: Calculus Teaching; Cognitive Root; GeoGebra; Differential Equations; Documentational Genesis.

Aprendizagem Interativa de Cálculo: integrando Raízes Cognitivas, Gênese Documental e GeoGebra

RESUMO

Contexto: Este artigo busca contribuir para o ensino de Cálculo no que diz respeito à seleção de recursos didáticos que considerem as dificuldades de aprendizagem. **Objetivos:** Defendemos a escolha de recursos apropriados que reflitam os resultados de teorias e as vantagens do uso da tecnologia. **Design:** Os construtos

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teóricos propostos por David Tall (1941-2024), como raiz cognitiva e retidão local, bem como a gênese documental introduzida por Gueudet e Trouche, são utilizados para fundamentar nossa abordagem aos conceitos de derivada e equação diferencial. **Coleta e Análise de Dados:** O estudo caracteriza-se como um estudo teórico em que são apresentados exemplos práticos. **Resultados:** Exemplos como a função “manjar branco” e o uso do GeoGebra são discutidos como ferramentas que promovem a compreensão de conceitos complexos por meio de visualizações interativas e dinâmicas. **Conclusões:** Este estudo enfatiza a importância de combinar teoria e prática no uso de tecnologias para facilitar a aprendizagem significativa de Cálculo, destacando a relevância dessas abordagens no campo da Educação Matemática.

Palavras-chave: Ensino de Cálculo; Raiz Cognitiva; GeoGebra; Equações Diferenciais; Gênese Documental.

INTRODUCTION

Calculus teaching has posed numerous challenges to teachers in their didactical efforts aimed at students’ meaningful learning. Various aspects have been highlighted, and searching for ways to overcome them presents another challenge that sparks the interest of researchers and teachers in their professional practice.

Mathematics, especially Calculus, has not evolved through an accumulation of novelties. There were coming and goings until the following aspects were stabilized: the definition of limit and of function, the conceptualization of number, among others. It was necessary to overcome epistemological obstacles to the constitution of its essential ideas, which were then supported by infinitely large and infinitely small sets.

In its early days, Mathematics was understood as the study of ultimate truths, of eternal reality, not immanent to nature and the universe, rather than a branch of logic and a tool for science and technology, as it is viewed today.

The skeptical approach of the Greeks, the science of Eudoxus (between 408 and 355 BCE) and Euclid’s consistent postulates revealed, however, that Mathematics could be useful in interpreting nature, alongside the Scholastic understanding that the Universe was organized and intelligible and the shift from a qualitative approach to motion and variation to a quantitative study. In their experiments, Nicholas of Cusa (1401-1464), Kepler (1571-1630) and Galileo (1564-1642) are heirs to the idea that Mathematics had an intuitive and experimental nature. However, in the 16th and 17th centuries, this image of architect of the universe was modified due to the need generated by practice.

Nevertheless, in the 19th century, the efforts to achieve a satisfactory foundation for Mathematical Analysis through the infinite drew conceptual criticism. Kant's (1724-1804) ideas encompassed the causes for this criticism, which, unlike Kant himself, no longer considered Euclid's postulates as categories of synthetic judgments, but just as premises. The criticism was based on the fact that said premises were chosen arbitrarily and, for this reason, could suppress evidence of the meaning of things.

At the end of the 19th century, the arithmetization of Mathematical Analysis embraced the concept of the infinite, considering that even though it transcends intuition in analysis, it could be inserted into Mathematics without implying that the logical consistency of study subjects should be abandoned. The conception that Mathematics is the science of quantity, space and number lost strength. The mathematical theory of continuity, which originated from experience and was considered by mathematicians, was transformed by mathematicians themselves into definitions that transcended sensory imagination. There is a certain dispute between formalists and intuitionists, but it is possible to accept that the "exactness of mathematical laws, [...] concepts are suggested, although not defined, by intuition thus easily accounts for the fact that the results of mathematical deductive reasoning are in apparent agreement with those of inductive experience" (Boyer, 2012, p. 3).

Calculus concepts, such as derivatives and integrals, result from aspects of nature, but their definitions are based on the abstract mathematical conceptualization of limit. These concepts emerged when the Greeks experienced difficulties in dealing with ratio and proportionality based on their intuitions, guided by numbers they considered discrete.

The results of investigations that revealed the cognitive impact of different semiotic representations of mathematical concepts, with non-trivial treatments and conversions, entailed new requirements for the teaching and learning of these concepts (Duval, 1995).

The fact that technologies play an essential role in all actions within society today obviously includes school. Digital resources specifically focused on education have been developed in a special way, causing alterations even in the epistemological references of their concepts. For example, we mention the case of Dynamic Geometry. Since the 1970s, Mathematics Education gained prominence in Higher Education, and Calculus teaching and learning were primarily considered by researchers in the field. Mathematics Education established itself as a more mature field of knowledge and became a source of data that could contribute to teaching and learning more consistently.

Today, there are theories in the field that analyze the treatment of challenges, although they are quite resistant. New technologies foster the development of new strategies for teaching, and theoretical constructs suggest pathways for understanding difficulties and obstacles to meaningful learning.

It is important to be aware of these historical epistemological elements and research results to realize that they impact Calculus teaching and, consequently, its learning.

In order to promote learning, teaching strategies should, therefore, take into consideration the complexity of the field, the integration between theory and practice and the new technologies that are available.

A recurring aspect in research studies in the field is the confirmation that there are advantages in teaching strategies that use digital resources and, if this use is based on theoretical constructs, results may be more consistent. To exemplify it, Macêdo & Gregor (2020, p. 8) say that "technology is an excellent tool if properly used, especially when mathematical software collaborates to the understanding of contents and to problem-solving through graphical tools". According to Santos *et al.* (2022, p. 110): "Digital Technologies are particularly justified due to their ability to enable visualization and motion through software, as well as to the actions that put this ability into effect, presenting graphs and figures in motion". Domingues *et al*. (2023, p. 297) add that "the role of visualization and experimentation with technologies is highlighted in the development of differential thinking among students".

In this article, we present digital resources that are in line with theoretical constructs whose specificities value the use of technological resources and the epistemological constitution of Calculus concepts. Here we refer, respectively, to documentational genesis, document and resource constructs from the Theory of Documentational Approach, by Gueudet and Trouche, and concept image, cognitive root and local straightness, proposed by Tall (1941-2024).

The following sections contain a tribute to Tall, who passed away recently; consideration on the theoretical foundations of the research that guided this article; two digital resources, R1 and R2, and examples. The article ends with a discussion on the potential impact of these resources on students' learning, providing suggestions for future research in this field.

DAVID TALL (1941-2024)

In this section, we would like to pay tribute to English researcher David Tall, who sadly passed away in 2024. In Author (2013), there is a brief bibliographical account of the author based on the data available on the researcher's personal website (http://www.davidtall.com/).

David Tall can be considered one of the most influential researchers in the field of Mathematics Education. He left a vast legacy that is worthy of celebration. His academic journey started in 1960, when he entered Oxford University to study Mathematics. After standing out as the best student in his class and receiving the Junior Mathematics Prize, Tall pursued a doctorate under the supervision of renowned mathematician Sir Michael Atiyah (1929-2019) and concluded his thesis in the field of Mathematics in 1967.

However, a series of pedagogical experiences and challenges related to communicating mathematical concepts at conferences made Tall reflect on Mathematics teaching. Then, he decided to focus his research on Mathematics Education, which marked the beginning of a bright career as a mathematics educator.

After his transition to the study of Mathematics Education (Tall, 1986), he contributed significantly to the field with his work on Advanced Mathematical Thinking and the application of digital technologies to teaching (Tall, 2002). Graphic Calculus software and his empirical research influenced trends in the 1980s by introducing innovations to Calculus teaching. He was a pioneer in using computer graphics to facilitate the understanding of complex topics and collaborated with renowned mathematicians and educators, such as Ian Stewart.

Besides his theoretical contributions, Tall established a close bond with Brazil. He visited the country several times, participated in conferences and collaborated with institutions like Universidade Federal do Rio de Janeiro, Universidade Federal de São Carlos, Pontifícia Universidade Católica de São Paulo, and others institutions.

Brazilian researchers who worked directly with the English researcher include Márcia Maria Fusaro Pinto, Victor Augusto Giraldo and Rosana Nogueira de Lima.

David Tall left an invaluable legacy by transforming the way we conceive the teaching of mathematical concepts, always in search of a more accessible and visual approach, which has left a significant mark in the field.

THEORETICAL FRAMEWORK

The theoretical framework that supports our contributions in this article includes the main theoretical constructs developed by Tall and by Gueudet & Trouche. These approaches provide a basis for the development and application of digital didactical resources to Calculus teaching, and they may facilitate the learning of complex mathematical concepts, such as differentiability and differential equations.

The theoretical constructs developed by Tall, cognitive root and local straightness enable a progressive approach to mathematical understanding. Cognitive root serves as a starting point for an intuitive development of formal knowledge, while local straightness facilitates the understanding of how the slope of a function at a point resembles a tangent line. They guided the development of the interactive activities proposed in this article.

The constructs developed by Gueudet and Trouche, elements of the Theory of Documentational Approach, refer to classroom actions related to how students and teachers construct resources for teaching and how they are used for learning. This theory emphasizes the idea that teaching resources are not mere tools for teaching, but a part of a dynamic process of knowledge construction. Through this framework, we explore the importance of creating and using resources like GeoGebra in order to foster an interactive and visual comprehension of concepts, integrating theory and practice in Calculus learning.

The Documentational Approach to Didactics (DAD)

The Documentational Approach to Didactics (DAD) is a theory within the scope of Didactics of Mathematics proposed by Gueudet and Trouche (2008), aiming to understand teachers' professional development through studying their interactions with resources conceived for teaching purposes (such as textbooks) and with those which are not intended for teaching (for example, a journal article). The use of these resources, both in the classroom and outside it, encompasses selecting, modifying and creating new resources, which results in what is called teacher's documentational work in the DAD.

The distinction between the available resources and the document created by the teacher with a didactical intention seems to be in line with the distinction adopted in the field of document engineering:

> The conceptual and technological framework that we explore is that of document engineering, which replaces the computational properties of the digital technology in question, and distinguishes calculable document resource, as a means, and calculated document, as a purpose. (Gueudet & Trouche, 2016, p. 7)

According to Christo (2022), the interactions between the teacher and the resources available for teaching are the fundamental core of DAD. Trouche, Gueudet and Pepin (2018) aimed to contribute so that the perspectives of artefact, instrument and instrumental genesis (constructs from the Instrumental Approach (Rabardel, 1995)) were replaced with resource, document and documentational genesis.

The resources or systems of resources available to teachers are not mere material resources such as a computer or a textbook. They also encompass a mathematical and a didactical component attributed or built by the teacher for specific class situations. Through the mathematical component, it is possible to observe the notions involved, the techniques and tasks proposed, or the activity guidelines. The didactical component shows the organizational elements of the development of teaching proposals, or the sequence of actions planned for the development of the class (Christo, 2022).

Gueudet and Trouche's (2009) documentational genesis is a process of transforming a resource into a document. Just like the Instrumental Approach (IA) (Rabardel, 1995), this process has two interrelated components called instrumentation and instrumentalization.

Instrumentation is the component that addresses the influence of harnessed resources, their possibilities, limitations, conditions and restrictions in teaching practice.

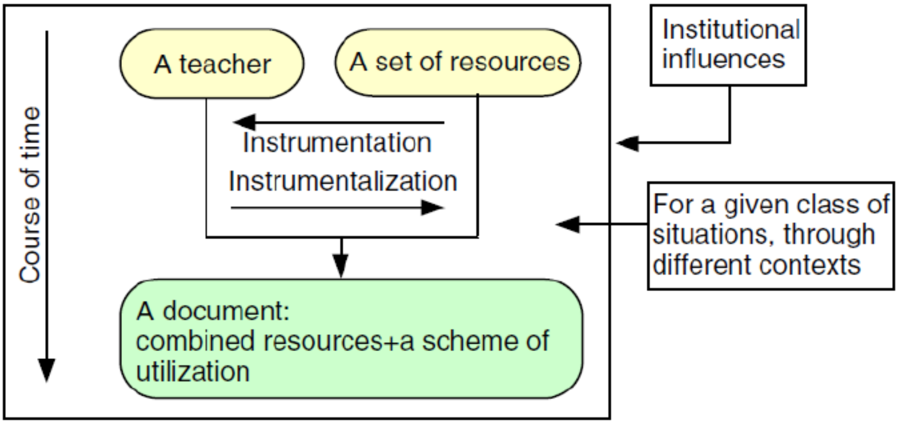
Instrumentalization focuses on the appropriation and (re)creation of resources by the teacher, modifying them in order to use them.

In the theoretical framework of DAD, when teachers interact with their system of resources, they perform a productive activity when they produce resources for teaching, and a constructive activity when they develop new knowledge.

Documentational Genesis is a complex process that encompasses the three dialectics displayed in Figure 1.

Figure 1

A Documentational Genesis Scheme. (Gueudet & Trouche, 2009)



The concepts of generic organizer and cognitive roots

In this section, we will present two theoretical constructs developed by David Tall to support the resources described in this article.

The concept of generic organizer (Tall, 1986) expands Ausubel's idea of "previous organizer", creating a learning environment where it is possible to manipulate examples and counterexamples of specific mathematical concepts. This environment should be interactive and focus on specific aspects of the concept, as is the case with manipulatives (for instance, Cuisenaire rods and Dienes blocks) and educational software. Tall warns us that a poorly structured generic organizer may lead students to abstract an incorrect property. According to the English researcher,

> If the generic organizer is used in an environment that is not properly controlled, then the student may abstract properties from the examples studied that are not part of the concept being modelled. As the human mind is a powerful pattern-detecting apparatus, patterns may be found that are not intended to be abstracted. (Tall, 1986, p. 83).

It may occur, for example, by using only continuous and differentiable functions; in this case, students might believe that all functions are like these, which reinforces the importance of including counterexamples, such as continuous yet non-differentiable functions. When one has no access to counterexamples, a property observed within a limited context may be incorrectly generalized. Tall called it the Principle of Generic Extension, which occurs in the following situation:

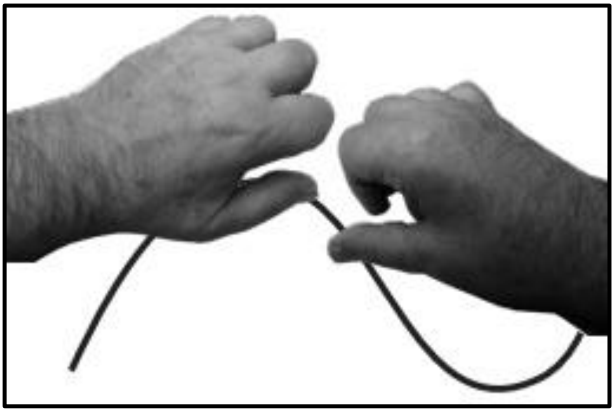
> If an individual works in a restricted microworld in which all the examples considered have a certain property, then, in the absence of counterexamples, the mind assumes the known properties to be implicit in other contexts. (Tall, 1986, p. 84).

The construction of a generic organizer requires using a cognitive root, which serves as an intuitive starting point for students. According to Tall (1989), a cognitive root is an "anchoring concept" that students understand easily while holding the potential for developing formal mathematical concepts. It may help in the transition to formal and abstract learning, and it should be carefully selected to avoid misunderstandings.

For instance, in the teaching of differentiable functions, the notion of local straightness can be used as a cognitive root, allowing students to visualize how a function resembles a straight line when magnified, as shown in Figure 2.

Figure 2

A small portion of a curve resembles a segment of a straight line. (Tall, 2013)



In the context of teaching derivatives, using local straightness enables the concept of limit to be implicitly considered in graphical visualization, whereas the derivative can be understood as the slope of the tangent line at each point of the graph.

Tall also suggests applying local straightness to the teaching of differentiable equations, allowing students to understand how to construct the graph of a function based on its known derivative. Moreover, using the slope field offers a tangible understanding of the behavior of solutions to differentiable equations.

Thus, cognitive roots facilitate an initial comprehension of concepts intuitively, preparing students for a later theoretical formalization. The resources presented, such as the exploration of the blancmange function, which illustrates the non-differentiability of continuous functions, serve as examples of cognitive roots that enable progressive learning.

DIGITAL DIDACTICAL RESOURCES FOR TEACHING AND LEARNING CALCULUS

This article presents the description of two digital didactical resources (R1 and R2) that were developed based on the theoretical constructs proposed by Tall and his collaborators, aiming at the formation of rich conceptual images[1] of the concepts involved.

R1 is a set of graphs of real functions of a real variable constructed in GeoGebra (Author, 2017) that allows teachers, through its available features, to explore the relationships of continuity and differentiability of these functions. Using software facilitates the desired exploration.

R2 enables the exploration of a solution to ordinary differential equations (ODE).

The R1 resource: the relationship between continuity and differentiability

R1 aims to harness conceptual images in the learner in order to consolidate the relationship between differentiability and continuity and to allow the reversal of mistaken conceptions constructed through usual and recurring examples provided in the classroom. It is formed by three examples:

[1] According to Vinner (2002, p. 68): “The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it also can be a collection of impressions or experiences”.

the first is the main one, while the other two are complementary, but equally important.

The first resource is a function called blancmange, a significant example for analyzing the relationship to be explored, because it is a continuous function at all points of its domain, yet non-differentiable at any of them. The idea here is to take to the extreme the fact that continuity does not imply differentiability.

That is to say, it is a very illustrative example that the T1 theorem[2] does not imply a reciprocal relationship.

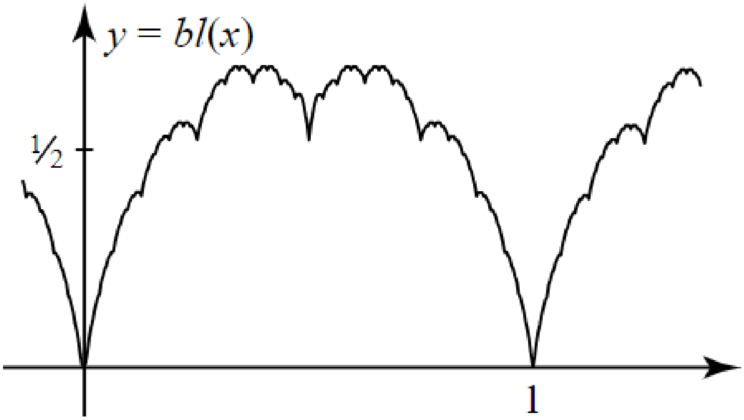
In general, the example presented to illustrate that T1 does not imply reciprocity is the modular function expressed by $h(x) = |x|$ at $x_0 = 0$. However, this fact may be overlooked as an isolated case among the many other examples of functions in which the reciprocal is valid. In light of this, providing other examples is important, and it is especially important to promote the study of functions that are continuous yet non-differentiable at more than one point in the domain.

The example considered by Tall is the function he named blancmange, in reference to a type of English dessert (Tall & Giacomo, 2000). This function was introduced by Japanese mathematician Teiji Takagi (1875–1960) in 1903 (Takagi, 1990) and illustrates an extreme case of rupture in the identification between continuous and differentiable functions. It is characterized by its wrinkled look, which contrasts with the smooth aspect of functions studied in Calculus courses (polynomial, exponential and trigonometric functions, for example). Figure 3 shows its graphical representation.

[2] T1: If a real function *f* of a real variable is differentiable at a point x_0 in its domain, then *f* is continuous at x_0.

Figure 3

Graphical representation of the "blancmange" function. (Tall & Giacomo, 2000)

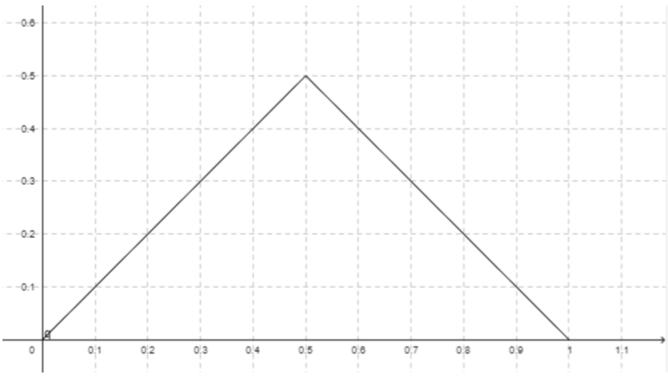


The graphical representation of the "blancmange" function is constructed through the series (s_n) of functions where the first term of the series is the function called "sawtooth" $s_1 = s_1(x)$ with:

$s_1: [0,1] \rightarrow \mathbb{R}$ and $s_1(x) = d(x, \mathbb{Z})$[3].

Figure 4

Graph of the function s_1 (Developed on GeoGebra)



The algebraic representation of the "blancmange" function is:

[3] $d(x, \mathbb{Z}) = inf \{ |x - z|, x \in [0, 1]$ and $z \in \mathbb{Z} \}$.

$$b(x) = \sum_{n=1}^{\infty} s_n(x)$$

Each term in this series is algebraically represented by the expression $s_n(x) = \frac{1}{2^{n-1}} s_1(2^{n-1} \cdot x)$. In Tall and Giacomo (2000) and Author (2017), it is demonstrated that function b is continuous and non-differentiable on (0,1). Series b is convergent at all points on [0,1], so the function is well defined on this interval.

This example can be used to foster discussions on the relationship between continuity and differentiability. It illustrates that interpreting graphical representations of functions may not be enough to understand their properties. It is necessary to highlight the importance of performing an analytical study when studying advanced mathematics topics.

According to Tall, it is beneficial to approach this type of example. Through it, the development of rich conceptual images among students regarding the concept of functions may be boosted, because the

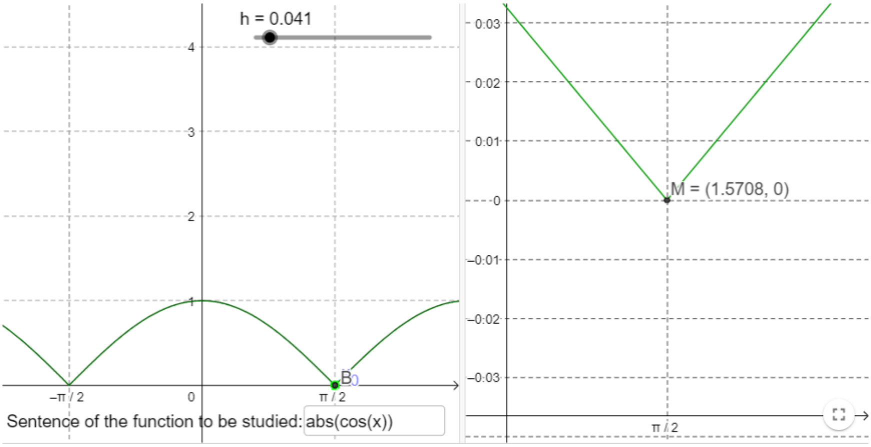
> [...] first course in calculus usually focuses on the regular functions given by a combination of the standard functions. This necessarily gives the impression that functions are usually differentiable, setting up the met-before that later ideas in mathematical analysis can be monstrous. An embodied approach, however, can offer insights as to what it means to be non-differentiable. (Tall, 2013, p. 314).

The reference to an embodied approach, as suggested by Tall, is based on the notion of local straightness. By harnessing this notion, it is possible to assume the non-differentiability of this function at a given point, that is, "the graphical representation of the function, in a neighborhood of this point, does not resemble a straight line at any level of magnification" (Author, 2017, p. 146).

For example, the real function $g(x) = |cos(x)|$ is not differentiable for any value of x such that $x = \frac{k \cdot \pi}{2}$, where $k \in \mathbb{Z}$. Figure 5 shows the graphical representation of g on the left. On the right, there is a magnified view of the graph of g in the neighborhood of $x = \frac{\pi}{2}$ by using 'MagnifyG'. This application is available on: https://www.geogebra.org/m/fgnghuqt.

Figure 5

Graph of the function $g(x) = |cos(x)|$ *and a magnified view of a neighborhood of point* $\left(\frac{\pi}{2}, 0\right)$ (Developed on GeoGebra).

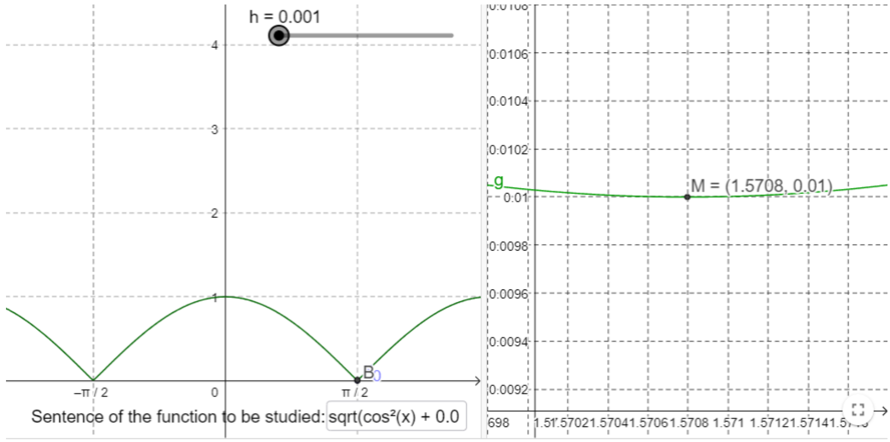


In the case of function g, a straight line will not be obtained, irrespective of the level of magnification of the neighborhoods around points on its graphical representation. It is important to consider this because there are functions that may not appear differentiable at certain levels of magnification of neighborhoods on their graphs, but they are indeed differentiable.

This is an adaptation based on a proposal by Tall and Giacomo (2000), where the function is *h,* with $h(x) = \sqrt{cos^2(x) + 0{,}0001}$. At a certain level of magnification in the GeoGebra Graphics, the graphical representation of this function resembles that of a function that is non-differentiable on $\left(\frac{\pi}{2}, 0\right)$, because it presents a "peak". However, when the level of magnification of the neighborhood of the point increases, it resembles a straight line parallel to the x-axis, indicating that the derivative of the function at this point exists and is equal to zero (which can be verified by calculating *h'* using the rules of differentiation).

Figure 6

Graph of the function $h(x) = \sqrt{cos^2(x) + 0{,}0001}$ *with a magnified view of the neighborhood of point* $\left(\frac{\pi}{2}, 0\right)$. (Developed on GeoGebra).



About using an example of a function that is continuous and non-differentiable for all values in the domain, we highlight two research publications: Tall and Di Giacomo (2000) and Oikkonen & Hannula (2022).

With this type of example, Tall and Di Giacomo (2000) indicate that it is necessary to use visual representations to generate perceptions of certain mathematical concepts. However, they are not sufficient, because it is essential to perform an analytical study of their differentiability, considering their algebraic representation of the limit function of the series of functions.

We consider that using these resources may contribute to shedding light on the frequent complaints made by advanced mathematics course students about rigor. What they "[...] often lack at the advanced level is the sense of *why* the abstraction was made" (Mamona-Downs & Downs, 2002, p. 168).

Tall & Di Giacomo (2000) emphasize that although graphical representations of functions are not precise, they are a pedagogical tool that may encourage discussion, improve the understanding of mathematical ideas and give meaning to the rigor that should be applied.

Oikkonen & Hannula (2022) also explore the example of the blancmange function. They use a visual approach to demonstrate the properties of this function, specifying the uniform continuity and indicating that the function is not differentiable at any point in the domain. The authors argue that the construction and comprehension of these functions may be enhanced by an approach that encompasses visual resources; that involves social-subjective and objective aspects of mathematical thinking.

By using the theoretical framework of the Three Worlds of Mathematics (Tall, 2013) – embodied, symbolic and formal – and expanding it with the addition of social-subjective and objective aspects, we have a comprehensive six-dimension structure (Oikkonen & Hannula, 2022). This approach highlights the interaction among different ways of comprehending mathematics, suggesting that teaching and learning complex mathematical concepts, such as conjecturing about functions that are continuous yet non-differentiable at any point, may benefit from a multifaceted perspective that adds visual and social elements to mathematical discourse.

Furthermore, an example of continuous and non-differentiable function was important for the historical development of the concept of function. Roque (2012) stated that the development of the concept originated from a process that evolved from an intuitive approach, linked to concrete problems, for the most abstract and rigorous definition we have today, based on the set theory. Initially, functions were found in physical and practical contexts, yet, with the advancement of mathematics in the 19th century, the field saw the emergence of "pathological" functions that challenged intuition, such as the Dirichlet (1805–1859) function and the continuous yet non-differentiable Weierstrass (1815–1897) function.

These two pathological examples brought contributions that led mathematicians from the 19th century to question and revise the definitions of function, continuity and differentiability, promoting autonomy in definitions based on abstract concepts, rather than relying on sensitive intuition or geometric perception.

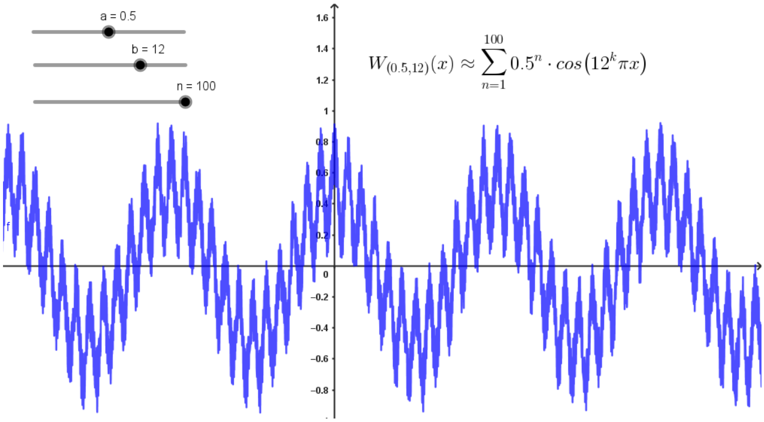
Weierstrass' continuous yet non-differentiable function is the series defined as

$$W(x) = \sum_{k=1}^{\infty} a^k \cdot \cos(b^k \pi x)$$

where $0 < a < 1$, b is any integer and $a \cdot b > 1 + \frac{3\pi}{2}$. In Figure 7, we present the partial sum of 100 terms of the function W for $a = 0{,}5$ and $b = 12$.

Figure 7

Graph of the partial sum of $\sum_{n=1}^{100} 0{,}5^n \cdot cos(12^n \pi x)$ (Developed on GeoGebra).



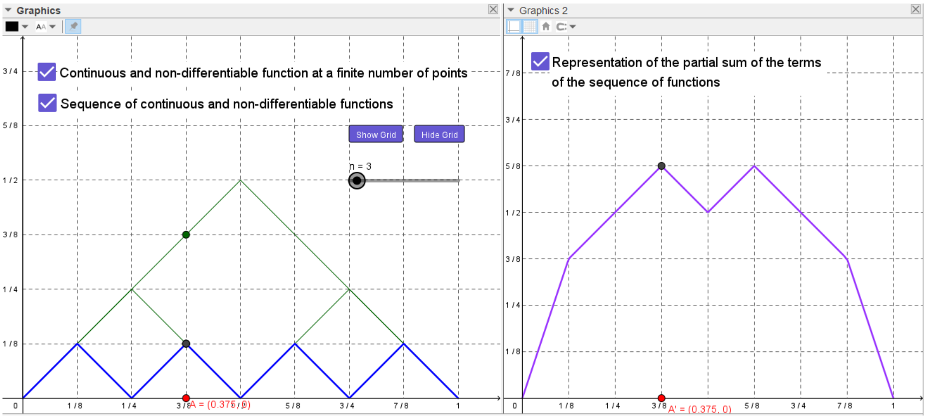
A proof of the continuity and non-differentiability of the function and other types of continuous yet non-differentiable functions can be found in Thim (2003).

In the case of the material presented in this section, an application was created in GeoGebra to display the terms and the partial sum of the series of functions that has the blancmange function as its limit.

This application is called ManjarBrancoG (available on https://www.geogebra.org/m/tn2ur4fx), which is an adaptation of David Tall's proposal, as shown in Figure 8.

Figure 8

ManjarBrancoG application (Developed on GeoGebra).



This application features the following aspects: in the Graphics window (on the left of Figure 8), there is a slider called n, ranging from 1 to 30, two buttons and two checkboxes to display/hide objects. The two buttons are used to display or hide the grid in both Graphics windows. The first box is called ‘Continuous and non-differentiable function at a finite number of points’, which presents the nth term of the sequence of functions whose limit is equal to the blancmange function. The second box is called ‘Sequence of continuous and non-differentiable functions’, which displays all first n terms of the sequence of functions.

The Graphics 2 window (on the right side of Figure 8) has a checkbox to display/hide objects called ‘Representation of the partial sum of the terms of the sequence of functions’, which presents the partial sum of the series of functions whose limit is the blancmange function.

An aspect of ‘ManjarBrancoG’ that affects both Graphics windows is point A. This point can be moved only along the x-axis, and all other points to be plotted will be related to it. The points displayed in the Graphics window are the images of the abscissa of point A for each real function in the sequence of functions.

In Graphics window 2, there are two points: the first one is on the x-axis and has the same abscissa and ordinate values as point A; the second point is equal to the image of the abscissa of point A under the function that results from the partial sum of the n functions in the sequence. These points were conceived so that users can analyze the ‘peaks’ of a function, which is an element of the sequence of functions, through the partial sum checkbox.

We understand that this application is in line with the theoretical construct of local straightness because it makes use of visual representations to explore the behavior of continuous yet non-differentiable functions at specific points, or at all points within a domain. The concept of local straightness encompasses the idea that, for a function to be differentiable at a point, the graphic representation in the neighborhood of this point should resemble a straight line when considerable magnification is applied. When it does not occur, then the function is not differentiable.

The R2 resource: a solution to an ordinary differential equation

R2 aims to explore the concept of solution to an ordinary differential equation (ODE) by harnessing the idea that the derivative at a point represents the slope of the tangent line to the function at this point and may offer a linear approximation of the function in its neighborhood. It involves the graphic construction of the solution to an ODE based on the slope of the tangent line at certain points, thus determining an outline of a solution to the equation[4], according to the initial conditions.

According to Tall, with the notion of local straightness, it is possible to present the concept of solution to a differential equation as the inverse problem of differentiation. In his words: “[…] (No, this is not integration!) The problem is this — if I know the gradient of a function at any point, how can I build up the graph that has that gradient?” (Tall, 2000, p. 14, adapted).

Furthermore, Tall proposes the following approach to the concept of solution to a differential equation, which

> [...] can be performed intuitively with little knowledge of the theory of differential equations. Yet it already carries in it the seeds of powerful ideas about possible existence theorems (that a typical first order differential equation will have a unique solution through each point), and following the changing direction will build up a global solution curve. By considering

[4] Consider the first-order ODE given by $\frac{dy}{dx} = f(x, y)$, a function $\phi = \phi(x)$ will be a solution to the ODE if $\frac{d\phi}{dx} = f(x, \phi(x))$.

> selected examples, it is possible to look at the wider view of what happens to a whole range of solution curves and to see their behaviour. In this way, an intuitive interface can provide advance organizers for formal theory, especially to those individuals who naturally build on visual imagery. (Tall, 2000, p. 212).

Besides, there is the attribution of the following embodied meaning to the form through which it is possible to outline a solution to one of these equations:

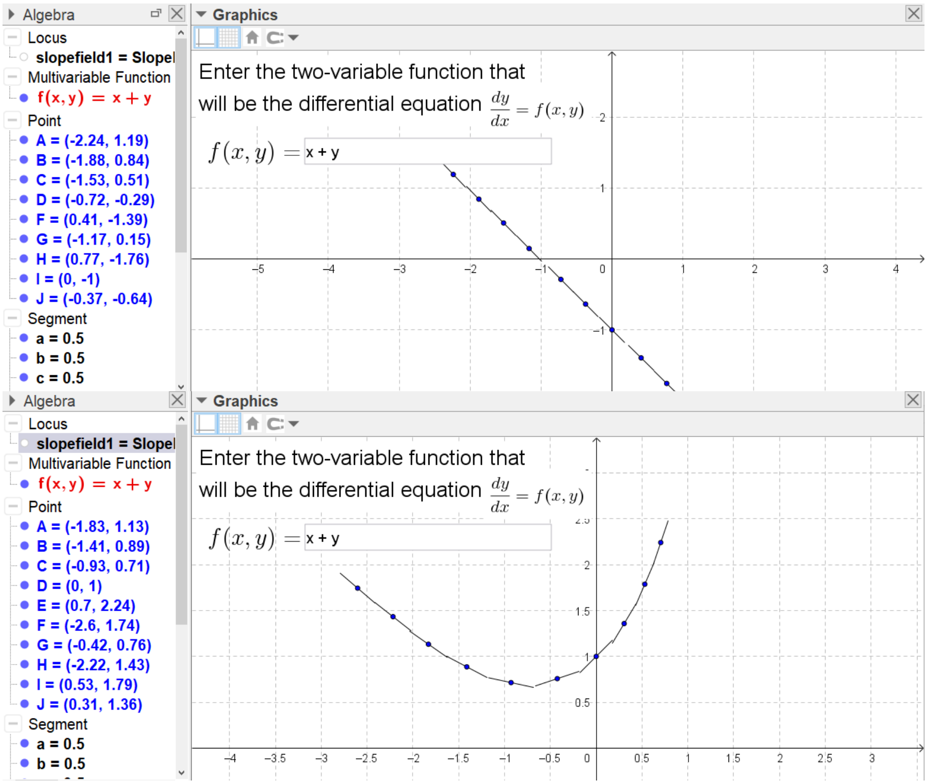
> If I point my finger at any point (x, y) in the plane, then I can calculate the gradient of the solution curve at that point as $m = F(x, y)$ and draw a short line segment of gradient m through the point (x, y) (Tall, 2002, p. 16).

R2 is an adaptation of a piece of software developed by Blokland, Giessen and Tall (2000 *apud* Tall, 2002) that simulates the concept of embodied meaning, as suggested by Tall and his collaborators (Available on: https://www.geogebra.org/m/mrz6eeqa). The application, developed in GeoGebra, enables the gradual construction of a solution curve of a differential equation by using segments that approximate the solution locally through the slope of the tangent line.

In Figure 9, we present a tool called EquacaoDiferencial. It uses a function of two variables, representing the ODE $\frac{dy}{dx} = f(x, y)$, and a point $A = (a, b)$ to create a line segment of length 1 with a slope equal to $f(a, b)$. The figure shows two examples of it being used for the equation $\frac{dy}{dx} = x + y$, under different initial conditions.

Figure 9

Using the EquacaoDiferencial tool for the equation $\frac{dy}{dx} = x + y$, *with two initial values:* $y(-1) = 0$ *and* $y(0) = 1$ (Developed on GeoGebra).



This tool allows the development of activities based on Tall's notion of embodied meaning, enabling users to explore differential equations through the graphical construction of segments that outline one of the solutions to the ODE. We suggest that the ODE used at the beginning of the process have, as solutions, functions that may be more familiar to students, such as straight lines, parabolas and hyperbolas.

In the example presented in Figure 9, there is an example of a differential equation whose solution curve changes according to the initial condition. For the equation $\frac{dy}{dx} = x + y$, the general solution is the family of functions $\{y_c\}_{c \in \mathbb{R}}$ given by $y_c = C \cdot e^x - x - 1$. For the initial condition $(-1, 0)$, the solution is the function $y_{-1} = -x - 1$, and, for the other initial condition (m, n), where the relationship $n = -m - 1$ does not hold, the curve is a function formed by the sum of an exponential function and a first-degree polynomial function.

Another example that can be used alongside the tool that has been presented is the equation $\frac{dy}{dx} = 1 - y$, whose general solution is the family of functions $\{z_c\}_{c \in \mathbb{R}}$, given by $z_c(x) = 1 + C \cdot e^x$. For the initial condition $(0, 1)$, the solution is the constant function $z_0(x) = 1$. For the other initial condition (m, n), where the relationship $n = 1$ does not hold, the curve is a function formed by the sum of an exponential function and a constant.

Monaghan et al. (2023) highlight the importance of geometric and visual approaches to facilitate the understanding of differential equations, because traditional analytical methods are often focused on exact solutions, which may be challenging for students (Rasmussen, 2001). Their research shows that many students have difficulty in perceiving that an ODE has a family of solutions and in distinguishing between numerical approximations and exact solutions. Monaghan et al. (2023) share that students

> [...] did not see solutions to DEs as functions, and that these can be presented graphically, understand equilibrium solutions as constant functions that satisfy the DE and appreciate that a numerical approximation and an exact solution are not the same. Others have found that even students who could find algebraic solutions for DEs "did not fully understand the related concepts, concepts, and they had serious difficulties in relation to these concepts" (Arslan, 2010, p. 873). (Monaghan et al., 2023, p. 219).

The approach proposed by Tall focuses on the graphic construction of solutions to the ODE, enabling students to explore these concepts through local straightness and visualization of the slopes of tangent lines at different points of the solution. By working with GeoGebra to draw graphs of solutions, students can see the difference between numerical and exact solutions in practice, addressing directly the difficulties described in this text.

In this subsection, we have presented a resource for teaching differential equations. We understand that it may enable students to visualize the outline of a solution curve through a set of segments, each with a slope corresponding to the derivative at the abscissa of a point. This is in line with the embodied approach proposed by Tall (2002) and suggests that by drawing these tangent lines, students may gain an idea of what the solution curve is, even though they do not know analytical methods for finding solutions to differential equations.

CONCLUSIONS

The digital resources presented in this article were developed in order to integrate Mathematics Education theoretical constructs into teaching practice, valuing the relationship between theory and practice. The concepts of resource, document and documentational genesis are theoretical constructs belonging to the Theory of Documentational Approach to Didactics, proposed by Gueudet and Trouche (2009), and the concepts of cognitive root and local straightness were introduced by David Tall. This combination facilitates students' conceptual development, more specifically, in the case of this article, regarding the concepts of derivative and solution to a differential equation.

The digital resources developed to explore the relationship between continuity and differentiability are directly in line with theoretical constructs developed by David Tall and his collaborators, especially the notion of local straightness. This type of resource harnesses the perception of how visualizing representations of mathematical concepts may help in understanding advanced ideas

The visualization of a set of segments illustrates the concept of local straightness applied to ODEs when a solution to the equation is constructed point by point based on the slope of a tangent line that belongs to the solution, and the slopes of these segments help in the qualitative study of the global behavior of this solution.

The results indicated that using software such as GeoGebra, when supported by Tall's theoretical constructs, may facilitate an intuitive understanding of mathematical concepts.

An important direction for future research may involve transforming these resources into documents developed based on teachers' instrumental genesis. Another path is to use these resources among various target audiences and contexts.

These research pathways are relevant because configurations of didactical resources may be transformed in different contexts, and schemes of utilization legitimize them. Configuring schemas means configuring structural forms of activities, a subject's invariant organization across a range of thematic situations, that is, operational invariants, theorems-in-action and concepts-in-action. It is essential because “the operationality of a concept encompasses a range of actions and schemes”; […] they are what “differentiates knowing how to do from the ability to make this knowledge explicit” (Franchi, 2012, p. 200-201, our translation).

Therefore, the development of research that facilitates the constitution of teachers’ documents for teaching derivatives and solutions to differential equations is a fundamental step toward reinforcing the value of this research, as well as contributing to its replicability.

We highlight the importance for future studies to deepen the investigation of the long-term impact of the resources presented in this article, especially among groups of students with different levels of familiarity with digital technologies. Furthermore, we suggest that new research studies should explore other technological tools for teaching Calculus and their possible contributions to the development of advanced mathematical thinking.

Again, we emphasize the importance of integrating theory and practice in Calculus teaching by using new technologies to facilitate meaningful learning. This study reinforces the relevance of research with this focus in the field of Mathematics Education, especially when it comes to topics that traditionally pose difficulties for students.

AUTHORS’ CONTRIBUTIONS STATEMENTS

MVA, SBCI participated in all phases of project development: preparation, participation in remote meetings and creation of activities. All authors actively participated in the discussions and reviewed and approved the final version of this work.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the authors for correspondence, upon reasonable request.

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