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Non-Classical Approaches to Logic and Quantification as a Means for Analysis of Classroom Argumentation and Proof in Mathematics Education Research[1]

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ABSTRACT

Background: While it is usually taken for granted that logic taught in the mathematics classroom should consist of elements of classical[2] propositional or first-order predicate logic, the situation may differ when referring to students' discursive productions. **Objectives**: The paper aims to highlight how classical logic cannot grasp some epistemic aspects, such as evolution over time, uncertainty, and quantification on blurred domains, because it is specifically tailored to capture the set-theoretic language and to validate, rather than to consider epistemic aspects. The aim is to show that adopting classical and non-classical lenses might lead to different results in analysis. **Design**: Nyaya pragmatic and empiricist logic, with Peircean non-standard quantification, both linked by the concept of free logic, are used as theoretical lenses in analysing two paradigmatic examples of classroom argumentation. **Setting and Participants**: excerpts from a set of data collected by prof. Paolo Boero from the University of Genoa during research activities in a secondary school mathematical class. **Methodology**: The examples are discussed by adopting a hermeneutic approach. **Results**: The analysis shows that different logical lenses can lead to varying interpretations of students' behaviour in argumentation and presenting proof in mathematics and that the adopted non-classical lenses expand the range of possible explanations of students' behaviour. **Conclusion**: In mathematics education research,

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[1] This is an extended and improved version of the author's contribution for CERME 12: https://hal.archives-ouvertes.fr/hal-03740184/document

[2] Classical logic is the logic where the law of the excluded middle ($A \vee \neg A$) and the law of non-contradiction ($\neg(A \wedge \neg A)$ hold, while non-classical logics are logics where at least one of these two characteristic properties does not hold. Examples of non-classical logics are the paraconsistent logic (the principle of non-contradiction holds only locally but not globally) and the intuitionistic logic (the law of the excluded middle does not hold and consequently also the double negation does not mean in general an assertion: $\neg\neg A \nvdash A$).

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the need to consider an epistemic dimension in the analysis of classroom argumentation and proof production leads to the necessity to consider and combine logical tools in a way specific to the discipline, which might differ from those usually required in mathematics.

Keywords: Logic; Non-standard quantification; Nyaya, Argumentation and proof; Set-theoretic language.

Abordagens não clássicas de lógica e quantificação como meio para análise da argumentação e prova em sala de aula em pesquisa de educação matemática

RESUMO

Contexto: Embora seja geralmente dado como certo que a lógica ensinada na sala de aula de matemática deve consistir em elementos da lógica proposicional clássica ou de predicados de primeira ordem, a situação pode ser diferente no que se refere às produções discursivas dos alunos. **Objetivos**: O artigo visa destacar como a lógica clássica não consegue apreender alguns aspectos epistêmicos, como evolução ao longo do tempo, incerteza e quantificação em domínios turvos, porque é especificamente adaptado para capturar a linguagem da teoria dos conjuntos e validar, em vez de considerar aspectos epistêmicos. O objetivo é mostrar que a adoção de lentes clássicas e não clássicas pode levar a resultados diferentes na análise. **Desenho**: A lógica pragmática e empirista de Nyaya, com quantificação não padronizada peirceana, ambas ligadas pelo conceito de lógica livre, são usadas como lentes teóricas na análise de dois exemplos paradigmáticos de argumentação em sala de aula. **Cenário e Participantes**: trechos de um conjunto de dados coletados pelo prof. Paolo Boero da Universidade de Gênova durante atividades de pesquisa em uma aula de matemática do ensino médio. **Metodologia**: Os exemplos são discutidos adotando uma abordagem hermenêutica. **Resultados**: A análise mostra que diferentes lentes lógicas podem levar a interpretações variadas do comportamento dos alunos na argumentação e apresentação de provas em matemática e que as lentes não clássicas adotadas ampliam o leque de explicações possíveis para o comportamento dos alunos. **Conclusão**: Na pesquisa em educação matemática, a necessidade de considerar uma dimensão epistêmica na análise da argumentação em sala de aula e na produção de provas leva à necessidade de considerar e combinar ferramentas lógicas de forma específica para a disciplina, que podem diferir daquelas usualmente exigidas em matemática.

Palavras-chave: Lógica; Quantificação não-padronizada; Nyaya; Argumentação e prova; Linguagem da teoria do conjuntos.

INTRODUCTION

In Mathematics Education (ME), logic can be considered from two different points of view: as a mathematical content that can be taught and learned at school or in university courses (e.g. D'Amore & Plazzi, 1992; Durand-Guerrier et al., 2012; Epp, 2003; Durand-Guerrier, 2020), and as a means for ME scholars to analyse students' discursive productions when they engage in proving, arguing or conjecturing (e.g. Arzarello et al., 2009; Durand-Guerrier, 2005, 2020; Durand-Guerrier & Arsac, 2003; Duval, 1991, 1992; 2007). This paper aims to help shed light on aspects related to the second way of considering logic in ME.

Literature review and rationale of the paper

While it is usually taken for granted that logic taught in the mathematics classroom should consist of elements of classical[3] propositional or first-order predicate logic (e.g., D'Amore & Plazzi, 1992; Durand Guerrier et al., 2012; Ferrari, 2010), the situation may differ when referring to students' discursive productions. Indeed, the kind of logic students spontaneously resort to when they conjecture, argue or provide proofs in the mathematics classroom is often difficult to capture with the formal instruments of propositional logic (Barrier et al., 2009; Durand Guerrier, 2005). On the other hand, quantification is helpful in better understanding reasoning because it allows us to consider predicates that can be true or false in reference to specific objects. However, quantifiers' consequent and explicit use is quite challenging in analysing students' reasoning. Indeed, students often use kinds of quantification or qualification that cannot be captured by a deductive argumentation scheme and by classical universal and existential quantifiers (Blossier et al., 2009). Some scholars propose natural deduction for first-order logic as a valuable means to reduce the distance between informal argumentation schemes and mathematical proof because of the possibility it offers to work on objects rather than on properties, reducing the need for the explicit use of quantifiers (Durand-Guerrier, 2005; Ferrari, 2010). Indeed, in natural deduction, there are quantifier introduction and elimination rules that reproduce "fairly standard mathematical practice" (Copi, 1965, quoted by Durand-Guerrier, 2005, p. 413). Durand-Guerrier underlines that natural deduction methods have been produced mainly for controlling validity in predicate calculus and that they can highlight quantification constraints that can lead to a lack of validity of a proof. In this

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[3] Classical logic is the logic where the law of the excluded middle ($A \vee \neg A$) and the law of non-contradiction ($\neg(A \wedge \neg A)$ hold, while non-classical logics are logics where at least one of these two characteristic properties does not hold. Examples of non-classical logics are the paraconsistent logic (the principle of non-contradiction holds only locally but not globally) and the intuitionistic logic (the law of the excluded middle does not hold and consequently also the double negation does not mean in general an assertion: $\neg\neg A \nvdash A$).

sense, the natural deduction is helpful to establish whether a proof is valid or not, but it is not tailored to capture the epistemic[4] reasons that might have produced a lack of validity.

On the other hand, in analysing students' reasoning, the primary goal is not necessarily the validation of a proof, as this is usually the case for mathematicians or could be the case when logic is taught in the classroom. For purposes related to the analysis of students' discursive production, the essential aspect is modelling the way students seem to reason, in an attempt to understand better the epistemic causes of failures or behaviour diverse from that expected by the teacher. Indeed, the awareness of such epistemic causes would increase understanding and thus the possibility of intervention by researchers in ME and teachers.

Hintikka's dialogical logic about game theoretical semantics is also studied to capture reasoning in ME (e.g., Arzarello & Soldano, 2019; Barrier et al., 2009; Blossier et al., 2009). A semantic game can be considered an interplay between the universal and the existential quantifiers, each governed by a player: the "Verifier" acts on the existential quantifier, and the "Falsifier" acts on the universal quantifier. The mathematical statement at stake is formulated in the form "$\forall x \exists y P(x, y)$." The Verifier has to exhibit for each object $x$chosen by the Falsifier from the universe of quantification an object y that satisfies the predicate $P(x, y)$. In this kind of logic, the concept of truth is related to the existence of a winning strategy for the Verifier; vice versa, the concept of falsity is associated with the existence of a winning strategy for the Falsifier.[5] Barrier and co-authors (Barrier et al., 2009) show that the proof process in a game-theoretic context consists of a dialogical exchange between semantic (action on objects) and syntactic aspects of proof construction. These authors use game theory, referring to Hintikka's (1996) concepts of *indoor games* (when the structure of the game allows for the use of objects through quantification) and *outdoor games* (when it is a matter of finding relevant properties and combining them in such a way that the result is a proof), in order to reduce the distance between informal argumentation and proof in the mathematics classroom. Arzarello and Soldano (2019) propose a model based on Hintikka's logic of

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[4] In the following, the term "epistemic" is used to refer to the acquisition of knowledge by the learning subject (e.g., the student), while the term "epistemological" is used to refer to knowledge acquisition in a scientific discipline, in this case ME.

[5] This concept of truth is based on the logical existence of a choice function that guarantees the existence of a winning strategy for the Verifier; it thus requires the axiom of choice (Hintikka, 2001).

inquiry (Hintikka, 1999) that is able to capture aspects such as the functional dependence between variables and parameters. This model has proved helpful in designing activities and applications in which students can approach the concept of proof in mathematics, producing and exploring conjectures.

Research shows that the approaches discussed above are complementary and can be considered globally consistent logical tools for didactic analysis in mathematics (Barrier et al., 2019). However, common to them is that they are classical or reducible to the classical approach.[6] Now, as Lindström's theorem shows, classical logic is intrinsically connected to set-theoretic language (Zalamea, 2021). In classical first-order logic, the variables referred to by the quantifiers range on sets representing the predicates' domains. One of the fundamental axioms of set theory[7] is the axiom of specification: given a set A and a formula φ(x), there exists a subset $B=\{a \in A: \varphi(a)\}$. This axiom is based on Frege's symmetry principle, according to which one obtains "an equivalence […] (locally, within the restricted universe A) between φ(a) (intensionality) and $a \in B$ (extensionality)" (Zalamea, 2009/2012, p. 324). If this axiom fails, both the law of the excluded middle (thus classical logic) and the standard use of quantifiers fail because it is not guaranteed that a property univocally determines a set.

On the other hand, the domain of reference of the statements during a learning process evolves and sets should become "variable" (Lawvere & Rosebgough, 2003) to grasp this evolution. Such sets can be captured by topoi in intuitionistic logic, considering an evolution over time, but not by classical sets. In this context, classical sets, and thus classical logic, could be viewed as special cases where time collapses into a moment. In this case, sets become fixed, the law of the excluded middle and the double negation hold, and thus also, classical logic holds.

Since classical first-order logic is tailored to capture classical set theory, restriction to set-theoretical language may not allow some kinds of reasoning to be recognised. Indeed, a truly "epistemic" rationality requires consideration

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[6] It can be shown that the truth concept in the game-theoretic semantic is equivalent to the Tarskian model of truth (Arzarello & Soldano, 2019) and thus to the truth conception in classical bivalent logic that follows the Aristotelian tradition.

[7] I refer to the Zermelo-Fraenkel axiomatic system with the axiom of choice (ZFC), as it is the standard axiomatic set–theoretic system within which mathematics usually is developed.

of indeterminacy, uncertainty and transformation over time, all futures out of range for set-theoretical language.

Occasionally, ME research also examines non-standard quantification as a historically accounted example of the difficulty in managing quantification in the classic sense (Blossier et al., 2009). These authors show that expert students (at the tertiary level) spontaneously use different kinds of quantification that often involve temporal aspects and a kind of variation of the variables that do not fit with the ∃∀-variation as it is known after the introduction of the axiom of choice. They also show that these kinds of quantifications can be historically accounted for in mathematicians' scientific production. Blossier and co-authors mention Bolzano's (link between constant and variable quantities) and Cauchy's (link between variable quantity and fixed limit) intuitive non-standard modes of quantification. They also account for the Peircean mode, highlighting that it does not rest on logical distinctions but is "inner to the individuum" (Blossier et al., p. 84). In our opinion, the Peircean non-standard approach is particularly interesting for the purposes of the present research due to its intrinsically epistemic features.

Research questions

This article aims to show that non-classical approaches to logic and quantification, which do not require set-theoretic assumptions, could help highlight epistemic aspects in the analysis of reasoning in mathematics classrooms that classical logic is not able to capture. In this way, (at least) novices' reasoning in ME, even if it does not fit classical logic, could be recognised as knowledge within a suitable logical frame rather than considered a lack of knowledge.

The following research questions are formulated to shed light on possible advantages of adopting non-classical logical lenses: Is there some evidence for the assumption that non-classical logical lenses lead to different results from classical ones in analysing classroom argumentation and proof? If so, what kind of advantages, if any, could be created by using non-classical logical lenses to improve knowledge acquisition in ME research?

The following paragraphs introduce some non-classical logical perspectives, and the hermeneutic methodological approach is outlined. Then, two paradigmatic examples are analysed and discussed. Finally, the results are summarised, the research questions are answered, and the conclusions are drawn.

THEORETICAL BACKGROUND

The theoretical approaches used to analyse the examples presented in the next paragraph from a non-classical perspective are very different from each other. The first is *Nyaya*, an empiricist logic rooted in Ancient Indian tradition (Sharma, 1962). It is suitable for framing a kind of reasoning close to everyday reasoning on objects that do not necessarily belong to well-defined domains. The second theoretical tool is the Peircean non-standard quantification (Peirce, 1960). At first glance, this kind of quantification might appear similar to the classical quantification. Still, it is genuinely epistemic because it considers *the way* the quantification is known rather than its grade of generality.[8] The third tool is *free logic* (Nolt, 2021). This kind of logic does not necessarily require a reference to a well-defined domain considered a closed set; the domain can be regarded as generally "the class of existing things" (Nolt, 2021, p. 1). Hereafter, these three theoretical tools are introduced and briefly discussed.

Nyaya and empiric rationality

In the Western mathematical tradition, the Aristotelian syllogism represents the basis of logical reasoning and only deductive syllogistic inferences are accepted for mathematical proofs. However, D'Amore (2005) shows that when dealing with proof, novices might spontaneously resort to a type of logic very different from the Aristotelian approach: one that is instead similar to the Indian Nyaya logic, a pragmatic and empiricist logic linked to perception. In this context, perception is not considered exclusively a matter of sensitivity. Indeed, in Nyaya-logic, the intellect is also considered a sense; thus, intellectual awareness is a kind of sensuous perception. In Nyaya logic, induction and deduction are closely interconnected within its "syllogism".

Furthermore, the use of examples is not only permitted but is actually expected by the argumentative model itself, and the "formal" and "material" aspects are closely intertwined within it (Sharma, 1962). For these reasons, the inferential model itself is conceived as a proof process of truth. According to D'Amore (2005), the Indian Nyaya philosophical school (1st century BC) awards prime importance to four sources of knowledge: testimony, analogy, perception and inference. Testimony includes everything that is considered worthy of faith. The analogy is a way of reasoning that allows us to define an

[8] This aspect will be discussed in the paragraph dedicated to the theoretical framework.

object based on its similarity to others; the correspondent to this concept in Aristotelian logic is the definition by the next genus and specific difference or the definition through an equivalence relation. Perception involves the (sensed) relationship between the object and the image that one has of it. The inference is what can be considered the Nyaya "syllogism" and has the following structure: (1) the Assertion (what is to be proved); (2) the Reason; (3) the Thesis (a general proposition followed by an example); (4) the Application; (5) the Conclusion.[9] Finally, one of the fallacies of the "right reasoning" in Nyaya is reasoning on non-existent objects, but, as stated above, from the Nyaya perspective, existence could also be a matter of intellectual perception.

Peirce's non-standard quantification

According to Peirce, quantification can be general, vague, or precise. He defines these categories "the three affections of terms, [which] form a group dividing a category of what Kant calls 'functions of judgment'" (Peirce, CP, 5.450).[10] *Generality* means the absence of distinction of individuals rather than validity for every individual, as is the case for the classical universal quantifier that quantifies over sets of individuals. This kind of quantification can be expressed in discursive language by words like *any*, *whatever*, etc. *Vagueness* means a specific type of existence that does not break the absence of distinction of individuals but states that there are suitable generic individuals that satisfy a particular property. This kind of quantification can be expressed by words like *some*, *certain*, etc. It is similar to the classical existential quantifier. Still, while the genericity of the latter rests on proof of independence from the choice of a specific individual, the former rests on *knowledge of the possibility* of choosing individuals that remain indistinct without a real actualisation. *Precision* means effective actualisation of possibility; the precise individual represents a rupture of the relationality that distinguishes the vagueness.

As with Hintikka's logic, the Peircean approach is a dialogic logic with a game-theoretic semantic (Pietarinen, 2019). Peirce talks about the "Graphist"

[9] An example of Nyaya-syllogism is the following: "(1) object *A* moves (statement); (2) because of a force applied to it (reason); (3) whenever we apply a force to an object, the object moves (general proposition); for example: if oxen pull a cart, the cart moves (example); (4) a force is applied to object *A* (application); therefore: (5) object *A* moves (conclusion)" (D'Amore, 2005, p. 27).

[10] Peirce's Collected Papers (CP) are quoted in the usual way: (Peirce, CP, volume number.paragraph number).

(who traces the signs corresponding to the statements on the sheet of assertion in his diagrammatic logic) and the "Interpreter" (to whom they are addressed). Still, the two roles can coincide in the same person. In this context, quantification is expressed as a degree of determinacy or indeterminacy of properties of a domain "under construction" that evolves during such a "dialogue."

As stated above, Peirce's quantification is epistemic but in a different manner from that of Hintikka. Indeed, as Zalamea (2021) shows, Peirce's logic can be captured by sheaf-logic, and sheaf-logic is intuitionistic. Thus, quantification in Peirce's logic does not require the axiom of specification, and the symmetry of Frege's abstraction principle generally fails. Furthermore, according to Hintikka (2001), intuitionistic logic is truly epistemic because the crucial notion: "is not knowing that, but knowing what (which, who, where, …), in brief, knowing + an indirect question, that is, knowledge of objects rather than knowledge of the truth" (p. 10) and this knowing-what-logic "cannot be analysed in terms of knowing that plus the apparatus of received first-order logic" (p. 11). Intuitionistic logic and its relations to classical logic can be suitably captured by categorical language (Caicedo, 1995) but not exclusively by a set-theoretical language. As Hintikka states, both his dialogical logic with game-theoretic semantics and intuitionistic logic are independence-friendly (IF) logics. IF logics permits free expression of quantifiers' dependence and independence, overcoming the usual syntactic constraints, but Hintikka's approach remains classical, while intuitionistic logic is an epistemic IF logic (Hintikka, 2001). In the Peircean "intuitionistic flavoured" logic, the independence of the quantifiers is expressed in terms of *knowledge* about their range rather than in terms of combinatorial concerns. Peirce's quantification creates a kind of logical distinction that is related to occurrences rather than objects: what is left to the Graphist (the "Speaker") or to the Interpreter (the "Listener") is the *mode of substitution* (precise, vague, or general) rather than its *grade of generality* (particular, generic, or general). The specificity of this kind of quantification is that it does not require the axiom of specification and Frege's abstraction principle: given a property (intensional aspect), the set that satisfies it (extensional aspect) needs not be actually "closed." Indeed, the Peircean logic to which this quantification refers can be considered a kind of *free logic*.

Free logic

In classical logic, singular terms denote objects belonging to the domain of quantification. This is not necessarily the case in free logic because singular terms need not denote existing things and might also refer to *unknown* objects (Nolt, 2010). According to Nolt: "Free logic is therefore useful for analysing discourse containing singular terms that either are or might be empty. A term is empty if it either has no referent or refers to an object outside the domain" (Nolt, 2010, p. 1).

The interplay of elements of the theoretical background

Nyaya logic is an example of empiricist logic where reasoning applies to single objects considered "existent" by the subject. This concept of existence does not necessarily imply that the object one refers to is an element of a set determined by a well-defined property (and thus well-known) but that it is perceived by the senses.[11] With its non-standard quantification, Peirce's logic can be considered an example of free logic, where the domain over which the quantifiers range is not necessarily a closed set. Indeed, vague quantification means that it is not necessarily known how the objects considered in reference to a singular term can be distinguished, but it is known that they exist.

The concept of free logic is helpful to support both the Nyaya approach and the Peircean non-standard quantification in reference to the future of the "existence" of the examined objects and their relation to the universe of the discourse. In this sense, Nyaya-logic and the Peircean non-standard quantification are compatible as approaches that need to refer to a free logic. They can be combined, at least at the basic level considered here.

METHODOLOGY

This theoretical investigation adopts a hermeneutic approach to text analysis (Palmer, 1969; Bagni, 2009). The methodology in the hermeneutic approach consists of interpretation intended as a dialectic back and forth between the meaning of the single parts of a text (oral, written etc.) and its global sense. The beginning of the interpretation is always based on the interpreter's presuppositions about the original context of the analysed text

[11] As stated above, it is important to bear in mind that in Nyaya the intellect is also a sense and thus allows a "sensuous" perception.

(cultural, historical, etc.). These presuppositions are needed to enter the hermeneutic circle. The concept of the hermeneutic circle, already characterised by Heidegger (1927/2017), has origins that date back to Schleieremacher (2000). It can be expressed as follows: to understand the whole (the complete text, be it verbal, oral or written etc.), the interpreter must understand its single parts, but it is only through understanding how the totality organises the single parts that these parts can be understood. In this sense, the presupposition about the text already anticipates the understanding of what is yet to be interpreted, but this apparent circularity in interpretation is necessary to produce knowledge. Indeed, if the interpreter remains open to the evolution and transformation of his or her own presuppositions as a consequence of the interpretative process, the hermeneutic circle can be transformed into a spiral (Bagni, 2009).

In the present context, the concept of personal space (Brown, 1996) of the protagonists (students and teacher) is used to frame the researcher's presuppositions in analysing the classroom transcripts. According to Brown, the personal space is the (virtual) space where "an individual sees him or her self-acting" (Brown, 1996, p. 120); it is made by all the aspects, interests, constraints and means that inform the subject's acting in a context and is a source for meaning because "the individual acts in the world he or she imagines to exist" (p. 121). In this research, the personal spaces mirror the students' and teachers' backgrounds, inferred by their surrounding cultural context while making mathematical statements or orchestrating mathematical classroom activities. The interpretation launches from these presuppositions and goes from the part (examples) to the whole (discussion) and vice versa, in a meaning-increasing process.

ANALYSES

In this paragraph, two examples are presented, analysed, and discussed using the theoretical lenses introduced in the theoretical background. The cases analysed can be paradigmatic examples from ME research, exemplifying the implications of an exclusive reference to classical logic and the implicit set-theoretic assumptions in the analysis of classroom argumentation and proof processes.[12]

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[12] Both excerpts stem from a set of data collected by prof. Paolo Boero from the University of Genoa during research activities in a secondary school mathematical class. The author came to know these data because they were used for a data-analysis-activity during a summer school for young researchers organised by the

As stated above, in the hermeneutic approach, the interpretative process starts with some presuppositions necessary to enter the hermeneutic circle. In the present research study, the presuppositions are represented by the elements that determine the students' and teachers' personal spaces. These personal spaces are characterised at the beginning of the sections, where the examples are analysed and discussed.

Example 1: Empiric rationality, Nyaya, and non-standard quantification

The first example introduced here is an argumentative text produced by a 15-year-old high school student. The question that s/he should answer is is it true that each number that ends with the digit 1 is a prime number (i.e., without divisors different from 1 and the number itself) or is it divisible by 3? The argumentation produced by the student is as follows:

> ***Student:*** *It is true. In fact, if we consider 11, 21, 31, 41, 51, 61, 71, etc., we see that they are all prime numbers or numbers divisible by 3. On the other hand, it is possible to reason also in general. If a number is prime and greater than 2, it ends with an odd digit (and 1 is just an odd digit). As for 3, among its multiples are 21, 51, 81, 111, 141, etc., all numbers ending on the right with the number 1. The thesis has been proven.*

The teacher considered this argumentation an example of the reversal of thesis and hypothesis and used it in another class to highlight this frequent cause of lack of validity.[13] Her approach is classical Aristotelian. As is well known, in the classical deductive model, proof starts from the hypothesis and ends with the thesis. On the other hand, in Nyaya-logic, the argumentation begins with the Assertion, i.e., with the statement to be proven, and this appears as a reversal of the thesis and hypothesis.

Italian Association for Research in Mathematics Education, in which she took part. The author thanks prof. Boero for authorising use of the data for purposes different from the original ones.

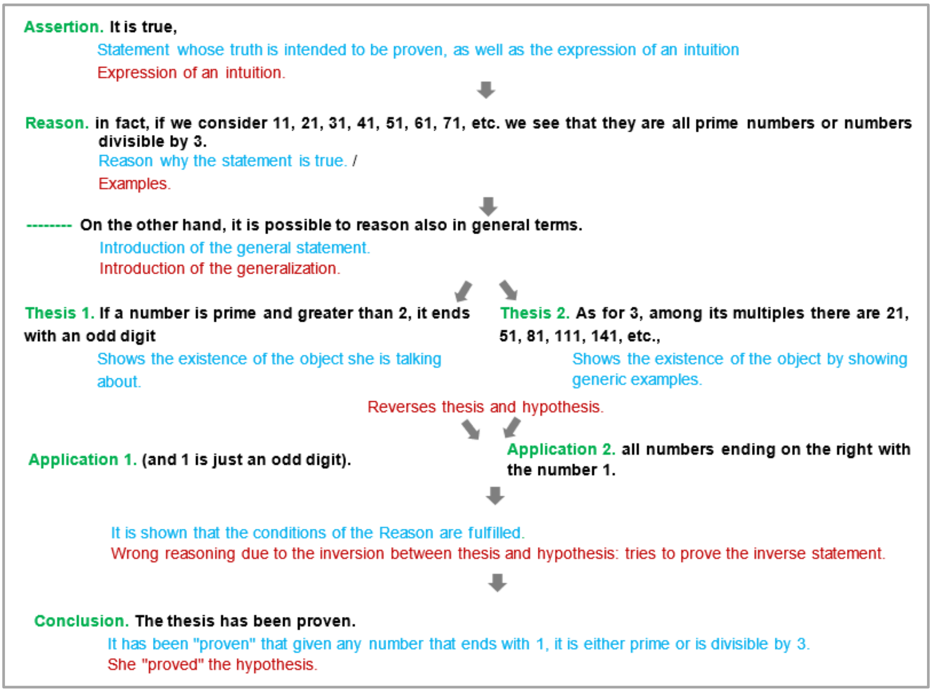
[13] The classroom discussion is examined in the second example.

Analysis and discussion of example 1

In Figure 1, the student's argumentation is analysed by adopting the classical Aristotelian approach and the Nyaya-lenses. Student's words are marked in **black bold**; the classification based on the Nyaya-scheme in green; the interpretation based on the Nyaya rationality frame in blue; and the one within the Aristotelian frame in red[14].

Figure 1

Analysis of student's argumentation adopting the Nyaya approach and the classical Aristotelian approach



[14] The Thesis and the Application of the Nyaya scheme are divided in two parts and the examples are missing.

The students' and teachers' personal spaces are framed by their unique backgrounds (professional and formative) and the classroom context.[15]

The student's personal space is characterised here by (1) the experience of the concept of proof in Euclidean geometry; (2) some initial explicit information about how a proof is made (thesis, hypothesis, general reasoning, no use of examples); (3) some elements of set-theoretic language in reference to number sets, without exploring quantification; (4) the interest in showing off one's own ability (the student was firmly convinced that their proof is a good one and s/he wants to prove the truth of the Assertion); (5) the constraint that the text is addressed to the teacher.

The teacher's personal space is framed by (at least) the following elements: (I) a valid proof starts from the hypothesis and ends with the thesis; (II) proof is deductive, and the use of examples means induction; (III) their spontaneous, implicit, or explicit, use of set-theoretic language as object language in mathematical contexts, due to their mathematical forma mentis. Both personal spaces are framed by the assumption that one "uses language in much the same way as everyone else" (Schulz, quoted by Brown, 1996, p. 121).

In the following discussion, the Nyaya-approach, in reference to the concept of empiricist rationality and Peirce's non-standard quantification are used as theoretical lenses. The fact that the student has not recognised that the statement is false does not matter because the focus is on their reasoning. From the teacher's "classical" point of view, the basis of the student's reasoning could be summed up as follows: The student tries to show that there is a partition of the set of numbers ending with 1 in two subsets: the set A, containing the prime numbers greater than 2 ending with 1, and the set B, containing the multiples of 3 ending with 1. However, s/he does nothing but show that the set of prime numbers ending with 1 is a subset of the set of numbers ending with an odd digit and that there are multiples of 3 ending with 1. Of course, in this way, s/he has not proved the existence of the supposed partition, but merely that there are two non-empty subsets of the sets A and B, thus reversing the thesis and hypothesis.

[15] As the analysed text was produced with research purposes completely different from that of this study, the information that was needed to reconstruct the teacher's and student's personal spaces was provided by prof. Boero himself. So, we know that the teacher graduated in the mid-1990s at the University of Genoa from a five-year graduation programme in Mathematics. In this instructional context, proof is based on classical Aristotelian approach and the object language is always set-theoretic.

Let us now eliminate references to sets in the mathematical sense, which do not belong to everyday reasoning: thinking of the number 3 does not necessarily mean thinking of it as a natural or rational number. We can think of it as an "object" in the same way that one thinks of a cup not as an element of the set of all cups but as an object that is perceived by the senses", i.e., as a perceived object. Adopting this point of view, we can see that the student lists some numbers ending with 1, followed by an ellipsis as if this list were to continue. The mathematically shaped mind might interpret this list as the representation of an infinite set. Yet this list is not necessarily an infinite set in the true sense; it probably represents indeterminacy or vagueness in Peirce's sense or, at most, potential infinity. Indeed, if the student reasons in terms of numeric sets, s/he should now try to prove the existence of the supposed partition, but s/he does not. However, if s/he does not reason in terms of numeric sets, what could s/he try to prove? Maybe given any number that ends with 1, that number is prime or is a multiple of 3. This reasoning is based on an interpretation of "each" (the universal quantifier) in the sense of "any," which has no meaning in classical first-order logic but represents generality in Peirce's sense. The student considers the first numbers listed as random cases (any) and finds that they have the required characteristics. This is an invalid generalisation in classical and Peirce's sense. The student has shown that some numbers exist that satisfy the property, and so s/he would be able only to quantify by recurring to a vague existence. This reasoning produces a sort of "fake" generalisation by induction. The student knows that generalisation by induction on single cases is not allowed and that s/he must produce a reasoning with general validity (the text is addressed to the teacher). What could be meant in the student's personal space by "reasoning that applies in general"? S/he seems simply to produce an existence proof, showing that the object being discussed actually exists in the sense of Nyaya logic and that it is precise in Peirce's sense: there are primes (greater than 2) ending with 1, and there are multiples of 3 ending with 1. Yet the proof is different in the two cases. In the first case, the student shows that the numbers whose existence s/he wants to prove are a particular case of other numbers, "defining" them by next genus (numbers ending with an odd digit) and specific difference (which end with 1). In the second case, the proof of existence is given by providing examples. However, s/he does not simply offer examples in common sense because s/he does not reason on particular multiples of 3 but on *some* multiples chosen by chance (thus, *vague* in Peirce's sense).

To sum up, adopting a non-classical approach in analysing the student's proof, at least the following hypotheses about students' conceptions could be formulated: (1) The student seems not to distinguish between vagueness (intended as randomness) and generality (intended as indeterminacy) in the Peircean sense; the awareness of this distinction seems to be a necessary condition to bridge the gap between everyday-rationality within an empiricist logic (Nyaya) and mathematical rationality; (2) The student's truth concept seems to be closer to that of an empiricist logic, i.e., to the idea of truth as existence (precise or vague), rather than to truth as general validity.

Example 2: Non-standard quantification within "blurred" domains

The second example refers to a classroom argumentative discussion led by the same teacher as in example 1 but in another class. A worksheet with the argumentation discussed in example 1 is used to introduce the argument, aiming to make students realise that this proof is invalid because the thesis and hypothesis are reversed.

Indeed, the first question the teacher asks the students is whether the proof is valid, but semantic aspects capture the students' attention: they detect two counterexamples (121 and 91) and state that it is false. The teacher brings the attention back to validity by asking about the reasoning on the worksheet.[16]

> [1] ***Teacher:*** *Let us see together if the proof is valid. In the meantime, what is to be proved?*
>
> [2] ***Student 1:*** *That if a number ends with 1, that is, if the last digit at the right is 1, then it is divisible by 3 or is prime.*
>
> [3] ***Student 2:*** *But that's not true; 121 is 11 times 11.*
>
> [4] ***Student 3:*** *and 91 too…*
>
> [5] ***Student 4:*** *Yes, 91 does not work either; it is 70 + 21, and it is divisible by 7, 7 times 13.*
>
> [6] ***Student 1:*** *For the examples he gave, it is fine, but for others, it does not work.*

[16] In the transcript UPPERCASE LETTERS are used for emphasised words, "…" for pauses longer than 5 seconds, and "[…]" for omissions in the transcription.

[7] ***Teacher:*** *Which is, instead, the reasoning made to prove that it is true that if the last digit on the right is 1, then the number is prime or divisible by 3?*

[8] ***Student 5:*** *The reasoning is that the multiples of 3 and the prime numbers end with 1.*

[9] ***Student 4:*** *No, SOME multiples of 3 and SOME prime numbers end with 1.*

[…]

[13] ***Teacher:*** *Well, it cannot work, but what is it that cannot work INSIDE the reasoning? If you had not discovered the counterexamples 91 and 121, the reasoning has been acceptable?*

[14] ***Student 6:*** *It seems not anyway, it is … I am close, but I do not know how to put it…*

[…]

[16] ***Student 8:*** *Maybe you want to say that … that for CERTAIN prime numbers or multiples of 3, things are going well because they end with 1, but this does not mean …*

[…]

[18] ***Student 3:*** *Yes, the reasoning says only that SOME prime numbers or multiples of 3 end with 1.*

[19] ***Student 9:*** *Even if ALL prime numbers or multiples of 3 should end with 1, there could be numbers that end with 1 and are not prime numbers or multiples of 3.*

[20] ***Student 6:*** *It is as if there is a reversal!*

[21] ***Teacher:*** *S6 said something important: "it is as if there is a reversal". It is an important idea!*

[22] ***Student 1:*** *The hypothesis and the thesis?*

[23] ***Student 6:*** *It seems to me to be a different matter!*

[24] ***Student 4:*** *To me, too, it is a matter … of numbers. Of sets of different numbers.*

[…]

[28] ***Student 9:*** *I will try to say it again; I do not know if it is OK: the multiples of 3 and the prime numbers are POSSIBLE numbers that end with 1, but these POSSIBLE numbers do not mean that they are ALL the numbers that end with 1.*

[29] ***Teacher:*** *I would say that is it.*

Analysis and discussion of example 2

In this example, the argumentation is carried out by a group of students. Nevertheless, one can state that the elements (1), (2), (3) and (5) of the student's personal space in Example 1 are also elements of the private spaces of these students because the cultural and formative backgrounds are the same. The element (4) of the student's personal space in example 1 is substituted by the following: (4') uncertainty about what validity means in a proof and how it can be accessed, besides the provision of counterexamples. This topic is addressed explicitly for the first time in this lesson. The teacher's personal space is the same as described in example 1 with the following addition: (IV) intention to focus the discussion on the lack of validity due to a reversal of thesis and hypothesis.[17]

The transcript shows that most punctuated words are related to some kind of quantification. The paraphrase of the input statement (line 2) and that of the reasoning whose validity students should judge (line 8) can be expressed in first-order logic as follows: $\forall x \in \mathbb{N}\ (A(x) \rightarrow (B(x) \vee C(x))$ (line 2) and $\forall x \in \mathbb{N}((B(x) \vee C(x)) \rightarrow A(x))$ (line 8). Apart from these two statements, the transcript shows the students' struggle with determining the domain of the statements expressed in the worksheet. Indeed, they intuitively detect a valid way to check the proof by considering the domains of the statement to be proved and of its contrapositive statement. Still, they are uncertain about how these two domains are related.

The non-standard quantification used by the students expresses the indeterminacy of the domain under consideration: *some*, *certain*, *not all*, *possible* numbers. For instance, as Student 4 (line 9) sums up the reasoning on the worksheet, s/he uses the term *some* as a vague existential quantifier in Peirce's sense because s/he knows that there are such numbers (the argumentation on the worksheet tells it), but their multitude is indeterminate;

[17] The points (4') and (IV) are based on a communication made by the researcher that collected the data.

s/he is not able to "close" epistemically a set with this property. In line 22, the teacher supports Student 6's intuition (line 20) that there is a reversal, meaning that the thesis and the hypothesis are reversed. However, the students' intuition is not a matter of hypothesis and thesis; it is a matter of "numbers," of "sets of different numbers" (lines 23 and 24): there are numbers that satisfy the thesis and hypothesis but also numbers that satisfy the thesis alone. By finally recognising the relationship between the domains of the statement to be proved and its inverse statement, the students could recognise that the reasoning on the worksheet is not valid. This is more than what the teacher wanted to highlight (reversal of thesis and hypothesis), although it is logically equivalent to it. Indeed, in the end, both students and teacher recognise their "idea of reversal" in the last formulation (line 28). However, one refers to the reversal of antecedent and consequent, and the other refers to relations between their domains.

To sum up, adopting a non-classical approach (specifically the Peircean non-standard quantification) in analysing example 2, the students' way of using quantification is suitably captured and described. This non-standard quantification expresses the epistemic uncertainty as vagueness related to what we called "blurred domains." Unlike in example 1, this argumentation produces an insight compatible with the teacher. Still, it does not fit the standard set-theoretic language and can be expressed in classical logic only by forcing the real epistemic constraints present in students' reasoning.

RESULTS

Following the hermeneutic approach in the discussion of the two examples, it was possible to show that the interpretation starting from presuppositions contained in the students' and teachers' personal spaces produce coherent frames for their behaviour and that the single parts fit well within the global sense of the interpretation.

In the first example, it was possible to explain the student's reasoning as knowledge in an empiricist rationality frame instead of "simply" as a lack of knowledge according to mathematical rationality. Indeed, adopting the classical approach, the only hypothesis able to explain the student's behaviour is the reversal of the thesis and hypothesis. However, this is nothing more than a diagnosis: it does not throw any light on possible interventions because it is not connected to the student's conceptions and does not tell us anything about the reasons behind the supposed error. On the other hand, the non-classical approach seems to delve deeper and provide more insight into possible causes of errors, linking them to the student's conceptions. This knowledge might offer greater possibilities for intervention by teachers and researchers.

In the second example, it was possible to capture students' reasoning suitably using Peirce's epistemic quantifiers. These quantifiers can express uncertainty about a domain of reference that is not fixed but evolves over time. Students' ways of reasoning brought much more insight into the cause of the lack of validity of the examined proof than the idea of reversal of thesis and hypothesis would do. However, this reasoning is unsuitable for classical quantification. It would need variable sets and also the concept of free logic. Indeed, while reasoning on the statement's validity, students do not know which objects (numbers) belong to the reference domain. This aspect can be captured by free logic but not by classical logic with its set-theoretic assumptions. Summing up, one can state that the students' way of reasoning was analysed by non-standard quantification. In this way, it was possible to show that this reasoning was different from the method used by the teacher, related to a classical approach. However, the two approaches led to logically equivalent results.

Further insights gained by adopting the non-classical lenses are as follows: (i) The novice's concept of truth might be related to the concept of the *existence* of the objects involved in the reasoning and not to a predicate that it might satisfy: A statement is true if the objects involved actually exist; this kind of existence could be "proven" on different levels: by showing one or more "exemplars" with the required characteristics; by referring to single objects as to randomly chosen examples, in a sort of genericity; by referring to a characterisation of the object in terms of next genus and specific difference; (ii) The concept of "reasoning that applies in general" might be related for the student to the production of a procedure of proof of existence, rather than to reasoning that applies to all cases and therefore to no one in particular; (iii) "Reasoning on objects" seems to imply often reasoning on blurred domains and thus non-standard-quantification.

All these aspects join some of the students' most recurrent difficulties concerning proof (Stylianides & Stylianides, 2017) and emerged thanks to the non-standard approaches in the analysis.

Summing up, the first research question can be answered in the affirmative because, in both examples, it was possible to show that there is evidence that non-classical logical lenses could lead to different results compared with classical lenses in analysing classroom argumentation and proof.

The second research question can also be answered positively. Indeed, in example 1, it was possible to broaden the range of hypotheses on possible causes for student errors. These hypotheses could help suggest areas of intervention focused on the student's previous conceptions rather than on their lack of mathematical knowledge. In example 2, the non-classic approach was helpful in better explaining students' reasoning and distinguishing it from the teacher's reasoning. Thus, it was beneficial to increase knowledge in ME research.

CONCLUSIONS

According to the hermeneutic approach, the process of interpretation aims to give a sense to the entire text, starting from the single parts and vice versa (Bagni, 2009). In both examples, the adopted viewpoint (Nyaya empiricist rationality, Peircean non-standard epistemic quantification, free logic) provides a coherent global frame for students' behaviour during the mathematical acts undertaken. On the other hand, the single aspects of students' reasoning become meaningful thanks to this global viewpoint expressed in the discussion.

The analysis shows clearly that students spontaneously resort to non-standard logics and non-standard quantification in Peirce's style. Still, it is important to stress that this study does not suggest substituting classical logic with alternative logics in ME. This would neither make sense nor be possible. As Barrier and co-authors show, classical logic (mainly natural deduction for first-order logic and Hintikka's dialogic logic) represents a coherent framework for analysing student behaviour (Barrier et al., 2021). However, these logical tools are too far removed from the tools students spontaneously resort to, especially when dealing with novices, due mainly to the underlying set-theoretic language.

Furthermore, logical tools are not neutral in reference to the information that can be detected or can remain hidden when using them for analysis. As stated at the beginning of this paper, it is essential to distinguish between logical tools suitable to be taught and learned, and logical tools suitable as a means for analysis in ME research, recognising the importance of considering epistemic aspects in the latter. However, classical logic is also needed to analyse student argumentation and proof productions, at least when the interest is focused on validating proof as the final product. In this sense, further research should examine possible shifts between different logical frames. In addition, the role of the relationship between metalanguage and mathematical object-language, not only in mathematics (Asenova, 2019) but also in ME, should be examined. What kind of object language should be used if the set-theoretical language cannot grasp all epistemic aspects? What would be a suitable logical model able to frame both the epistemic and the strictly mathematic approaches to logic and quantification, as needed in ME research?

These questions lead to yet a further issue: the question related to mathematical objects specific to ME research (Asenova, 2021). Indeed, a model of logic that responds to the particular epistemic needs of ME research would be a logical and mathematical means specific to this discipline because it arises from its particular methodological needs and epistemological constraints. The issue discussed in this paper is thus also an issue that closely concerns the epistemology of ME research.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the corresponding author, MA, upon reasonable request.

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