



# Affine Functions and Mixed Problems in Elementary School

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## ABSTRACT

**Background:** The understanding of the concept of function is complex for students of Basic Education to. The National Common Curriculum Base (BNCC) recommends that ideas of function, such as regularity, pattern generalisation and proportionality, be studied from the initial years of elementary school. **Objectives:** We aim to analyse function ideas mobilised by 5<sup>th</sup>-grade students when solving mixed problems, that is, problems involving operations of addition or subtraction and multiplication or division. **Design:** This study used a qualitative participant approach. **Setting and participants:** Thirteen elementary school 5<sup>th</sup> graders attending a rural school participated in the research. **Data collection and analysis:** The six groups of students solved four mixed problems implemented by the researcher during class time. The analyses were based on the conceptual fields theory, from audio recordings of the groups' dialogues, their written productions and through the researcher's notes. **Results:** The analyses show that the six groups of students expressed the ideas of function *correspondence, dependence, regularity, variable, proportionality, and affine function modelling*, and two groups expressed the idea of *generalisation*. Mixed problems can be associated with *affine function* ideas and can be solved by students since the initial years. **Conclusions:** Based on the results and on the conceptual fields theory, we argue that mixed problems should be proposed from the initial years so that ideas of function are appropriated and deepened by students during the school process.

**Keywords:** Conceptual fields theory; Mixed problems; Additive structure; Multiplicative structure; Initial years.

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## Ideias de função afim e problemas mistos nos anos iniciais do ensino fundamental

### RESUMO

**Contexto:** O conceito de função é complexo de ser compreendido por estudantes da Educação Básica. A Base Nacional Comum Curricular (BNCC) recomenda que ideias de função, tais como regularidade, generalização de padrões e proporcionalidade, sejam estudadas desde os Anos Iniciais do Ensino Fundamental. **Objetivos:** Busca-se analisar ideias de função mobilizadas por estudantes do 5º ano ao resolverem problemas mistos, ou seja, problemas que envolvem as operações de adição ou subtração, e de multiplicação ou divisão. **Design:** Este estudo utilizou uma abordagem qualitativa do tipo participante. **Ambiente e participantes:** Participaram da pesquisa 13 estudantes de uma turma do 5º ano do Ensino Fundamental de uma escola do campo. **Coleta e análise de informações:** Os seis grupos de estudantes resolveram quatro problemas mistos, implementados pela pesquisadora em horário de aula. As análises ocorreram com base na teoria dos Campos Conceituais, a partir de gravações em áudio dos diálogos dos grupos, de suas produções escritas e por meio de anotações da pesquisadora. **Resultados:** As análises mostram que: as ideias de função *correspondência*, *dependência*, *regularidade*, *variável*, *proporcionalidade* e a *modelação da função afim* foram manifestadas pelos seis grupos de estudantes, e a ideia de *generalização* foi manifestada por dois grupos. Problemas mistos podem ser associados a *ideias de função afim*, e podem ser resolvidos por estudantes desde os Anos Iniciais. **Conclusões:** A partir dos resultados e com base na teoria dos Campos Conceituais, defende-se que problemas mistos sejam propostos desde os Anos Iniciais para que ideias de função sejam apropriadas e aprofundadas pelos estudantes durante o processo escolar.

**Palavras-chave:** Teoria dos Campos Conceituais; Problemas Mistos; Estrutura Aditiva; Estrutura Multiplicativa; Anos Iniciais.

### INTRODUCTION

The concept of function is essential for mathematics and serves as the basis for various scientific and everyday situations. It was created by scientists and philosophers searching for explanations of natural-cause phenomena, the so-called “cause-effect,” such as the vaporisation of water (Caraça, 1951; Nogueira, 2014).

Studies (Rezende, Nogueira & Calado, 2020; Pavan, 2010) have found that the concept of function is not easy for students to understand. This may be due to its construction throughout the history of mathematics, as “[...] it took over 20 centuries of experiences, discoveries and disparities for this concept to

be formalised as it is currently conceived by the Mathematics community” (Calado & Rezende, in press, p. 2).

According to the National Common Curriculum Base – BNCC (Brasil, 2018), the concept of function must be officially studied in the 9<sup>th</sup> grade of elementary school and further in high school. This document proposes that ideas pertaining to functions, such as regularity, generalisation of patterns, and proportionality, must be presented from the early years, implicitly and without involving letters (Brasil, 2018).

Research on elementary school (Pavan, 2010; Silva, 2021; Rodrigues, 2021) has shown that multiplicative situations allow primary students to manifest basic ideas related to functions. Based on Caraça (1951) and Nogueira (2014), we assume that these basic ideas are common to all types of function, namely: correspondence, dependence, variable, regularity, and generalisation.

The present study assumes that a concept is understood from different situations experienced throughout the educational process, in connection with other concepts, properties, theorems, and situations interconnected, in what Vergnaud (1996a; 2009b) calls the conceptual field.

Multiplicative situations and additive situations were systematically studied by Gérard Vergnaud, who established two conceptual fields –that of additive structures and that of multiplicative structures, which present classes of well-defined situations that demand from students different schemes (organisation of the activity by the subject) for its resolution.

Besides additive and multiplicative situations, Vergnaud (2009b) defines mixed problems, which require at least one operation of the additive conceptual field (addition or subtraction) and at least one operation of the multiplicative conceptual field (multiplication or division) to be solved. Therefore, mixed problems can be solved by elementary school students, and textbooks at this level of education even propose them (Rodrigues & Rezende, 2021).

Due to their structure that involves an addition/subtraction operation and a multiplication/division operation, certain mixed problems allow the modelling of the affine function,  $f(x) = a \cdot x + b$  (Miranda, 2019).

Bearing in mind the mathematical language expected for elementary school, without algebraic formality, the present study elaborated four simple proportion and composition of measures mixed problems, which were solved by thirteen 5<sup>th</sup>-grade students. When proposing the situations to students, we

intended to answer the following research question: *What notions of function are mobilised by 5<sup>th</sup>-grade students when solving simple proportion and composition of measures mixed problems?*

For that, we used the theory of conceptual fields as a basis, which directed us to look at the schemas and operational invariants manifested by the students when solving the proposed situations. Following this section, we present the theoretical framework, methodological procedures, analyses, and main results of the study.

## **THEORETICAL FRAMEWORK**

The theory of conceptual fields assumes that knowledge is organised into conceptual fields, which students will understand over time through experience, maturity, and learning (Vergnaud, 2009a). To grasp a particular concept, students must experience various situations that enable the development of new schemes.

The concept of scheme is associated with the invariant organisation of activities by the subject. It has four components: a goal or several subgoals and anticipations; rules of action, collection and control of information; operative invariants (concept in action and theorem in action); and possible inferences (Vergnaud, 2009a). The organisation of gestures, whether of a baby, a child or an adult subject, contains the same components as a scheme: an objective, sequencing, coordination of movements of different body parts, and the identification of material objects and their properties (Vergnaud, 2009a).

One of these components, the operative invariants, gets attention from Vergnaud; it is about implicit knowledge manifested in the subjects' responses. There are two types of operative invariants: theorems in action and concepts in action. Theorems in action are knowledge in the form of propositions that can be true or false, and concepts in action are the concepts manifested by subjects through the theorems in action; they cannot be true or false; they are simply relevant or not for the situation at hand (Vergnaud, 2009a).

Theorems in action and concepts in action can also be seen as “[...] mathematical relationships that are taken into account by students when they choose an operation, or a sequence of operations, to solve a given problem” (Gitirana et al., 2014, p. 22). Between 8 and 10 years old, children understand that if a number of objects for sale is multiplied by 2, by 3, by 4, by 100, or by any other number, its price will be 2, 3, 4, or 100 times greater. This knowledge

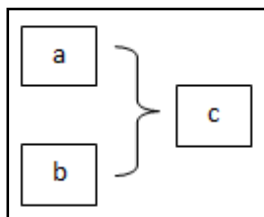
can be expressed by a true theorem in action: If  $n, x \in \mathbb{N}$ , then  $f(nx) = nf(x)$ , where  $x$  represents the number of objects,  $n$  is any number, and  $f$  is the relationship between the objects and the number (Vergnaud, 1996a).

Regarding the conceptual field, its main input is the set of situations that give meaning to the concept (Vergnaud, 2009a). The term concept is essential in theory and is defined from a psychological point of view. This is the set  $C$  composed of the triad  $C = (S, I, R)$ , in which:  $S$  is the set of situations that give meaning to the concept (the reference);  $I$  the set of operative invariants that structure the forms of organisation and operation of the schemes (meaning); and is  $R$  the set of linguistic and symbolic representations that are related to the representation of the concept (the signifier) (Vergnaud, 2009a).

Two conceptual fields were explored and structured further by Vergnaud: additive and multiplicative structures. The conceptual field of additive structures is composed of six classes of situations: composition of two measures into a third; transformation of an initial measure into a final measure; comparison relationship between two measures; composition of two transformations; transformation of a relationship; and composition of two relations. These situations require one or more additions and/or subtractions for their resolution.

### Figure 1

*Relational scheme for the composition of measures* (Vergnaud, 2009b)



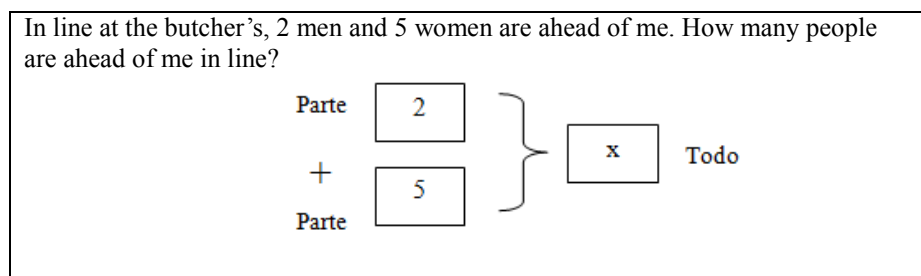
For each class of situations, Vergnaud establishes relational schemes that help us analyse their structure and classification. Considering that the composition of measures class is the one contemplated in the instrument used for data collection in this study, we present the details of this class below. Composition of measure situations are those that involve part-whole, “[...] joining one part with another to obtain the whole, or subtracting a part from the

whole to obtain the other part” (Magina et al., 2001, p. 25). Three measures are involved in this class, and its relational scheme is given in Figure 1.

Numbers  $a$ ,  $b$  and  $c$  are positive numbers, and the variations of this class are related to the extent to which one wishes to discover one of the parts, or the composition of measure, the whole (Miranda, 2019). Figure 2 presents a problem of this class.

### Figure 2

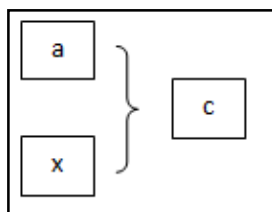
*Diagram of the composition problem (Magina et al., 2001, p. 25)*



This is an example of a composition problem in which, knowing the parts, it is possible to find the composition of measures, with “men” being one part of the problem and “women” being the other part; the sum of these two parts, men and women, form the whole. These situations can be represented by the equation  $a + b = x$ . For the problem in Table 1, we have the following equation:  $x = 2 + 5 = 7$ .

### Figure 3

*Sagittal scheme of a composition of measures (based on Vergnaud, 2009b)*



Another variation in the composition of measures class can be found in situations where, knowing one of the parts and the composition, the other elementary measure can be found. The equation representing this situation is  $a + x = c$  ou  $x = c - a$ , represented by the relational scheme presented in Figure 3 (Miranda, 2019).

Thus, there are two variations for the composition of measures class: the unknown composition and one of the unknown parts.

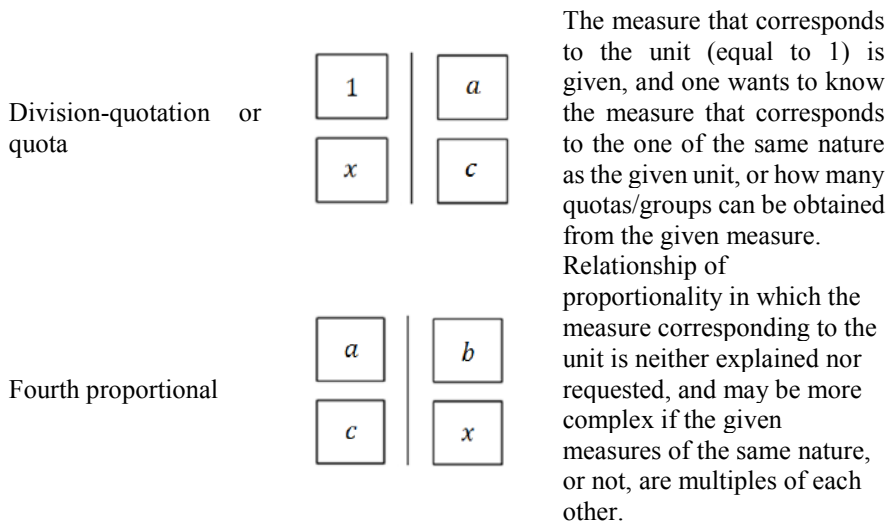
The conceptual field of multiplicative structures involves situations that can be solved through multiplication and/or division operations. It is organised into five classes: isomorphism of measures or simple proportion; multiplicative comparison; a single space of same-nature measures; product of measures or Cartesian product; bilinear function or double proportion; and multiple proportion (Vergnaud, 2009b; Gitirana et al., 2014).

The class of situations called *simple proportion* is the one contemplated in the instrument used in this study. For this reason, Table 1 shows the sagittal schemes related to the variations for that class.

**Table 1**

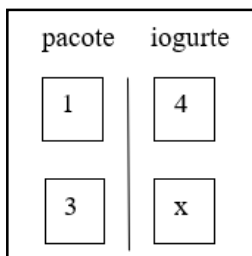
*Relational scheme of multiplicative type problems* (Miranda, 2019, p. 60, based on Vergnaud, 1993 and Gitirana et al., 2014)

Problem class	Relational scheme	Description		
Multiplication – one-to-many	<table border="1" style="display: inline-table; vertical-align: middle;"><tr><td>1</td></tr></table>   <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td><math>a</math></td></tr></table>	1	$a$	The measure has to do with the unit. It is given (unit 1), and one wants to know the value that corresponds to the second measure of the same kind as the unit.
	1			
$a$				
<table border="1" style="display: inline-table; vertical-align: middle;"><tr><td><math>b</math></td></tr></table>   <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td><math>x</math></td></tr></table>	$b$	$x$		
$b$				
$x$				
Division– partition or distribution	<table border="1" style="display: inline-table; vertical-align: middle;"><tr><td>1</td></tr></table>   <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td><math>x</math></td></tr></table>	1	$x$	The correspondence between two measures of different nature is given, and one wants to know the measure that corresponds to the unit.
	1			
$x$				
<table border="1" style="display: inline-table; vertical-align: middle;"><tr><td><math>b</math></td></tr></table>   <table border="1" style="display: inline-table; vertical-align: middle;"><tr><td><math>c</math></td></tr></table>	$b$	$c$		
$b$				
$c$				



**Figure 4**

*Sagittal scheme of the simple proportion class (Vergnaud, 2009b, p. 240)*



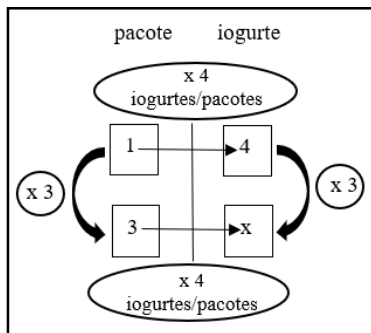
Simple proportion situations refer to a quaternary proportional relationship of the same measure, two by two and of the same nature. These relationships can be analysed in two ways: the vertical analysis and the horizontal analysis. For example, for the following problem: “I have 3 yogurt packs. There are 4 yogurts in each pack. How many yogurts do I have?” (Vergnaud, 2009b, p. 239), we have the sagittal scheme in Figure 4, both vertical and horizontal.



In the example, measures 1 and 3 represent the number of packs, and 4 and  $x$  represent the number of yogurts, i.e., they are two by two of the same nature. Figure 5 presents the horizontal and vertical analysis of this problem.

**Figure 5**

*Sagittal scheme of the simple proportion class (Vergnaud, 2009b, p. 243)*



According to Vergnaud (2009b), there are two ways to find the value of  $x$ , “[...] the first is to apply the dimensionless operator  $\times 3$  to the quantity 4 yogurts. The second, in applying the function  $\times 4$  yogurt/pack to the quantity 3 packs” (Vergnaud, 2009b, p. 244).

In addition to additive and multiplicative problems, Vergnaud (2009b) presents *mixed problems* as those that, for their resolution, involve at least one of the operations of the additive field, i.e., addition/subtraction; and at least one of the operations of the multiplicative field, i.e., multiplication/division (Vergnaud, 2009b).

Considering that mixed problems include at least one addition operation and one multiplication operation, one can suppose that mixed problems can be classified based on the classes of additive and multiplicative fields. Thus, it is assumed that there are at least 30 classes of mixed problems, since the additive and multiplicative fields have six and five classes, respectively (Miranda, 2019). Among the possible classifications for mixed problems, we consider the following for the present study: simple proportion (referring to the multiplicative field) and composition of measures (referring to the additive field).

As an example of a mixed problem: “Elisa bought a sewing machine and paid as follows: a down payment of BRL 250.00 and 3 more instalments of BRL 275.00 each. How much did she pay for the machine?” (Dante, 2017a, p. 137).

The example is a *one-to-many* multiplication simple proportion and composition of measures *with unknown part* mixed problem. To find the total value of the sewing machine, one must find out how many *reais* the value of the three instalments will be, and add it to the value of the down payment. Finding the value of the three instalments can be represented by a relational scheme defined by Vergnaud (2009b). The relational scheme in this problem refers to a quaternary relation of one-to-many multiplication of simple proportion.

Once the value of the three instalments has been found, i.e., BRL 875.00, all that is needed is to add it to the amount of the down payment to find the total paid for the sewing machine. The relationship for this final part of the resolution indicates a ternary relationship of composition of measures in which the composition is unknown, with BRL 875.00 referring to one measure and BRL 250.00 to the other measure.

Having presented the theoretical framework, the following section describes the development of the study and the analysis of the data produced by the students.

## STUDY DEVELOPMENT AND DATA ANALYSIS

This study<sup>1</sup>, of a qualitative nature, was developed in the second semester of 2021 with thirteen 5<sup>th</sup>-grade students attending a rural elementary school located in the northwest of the state of Paraná, Brazil.

Four mixed problems were developed for data collection. They all fit the classes of *simple proportion* and *measures composition*, whose didactic variables were carefully selected. According to Bittar (2017, p. 103), didactic variables are “[...] elements of the situation that, when changed, lead to changes in the students’ resolution strategies”.

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<sup>1</sup>All legal procedures were followed for the development of this research, and it was approved by the Research Ethics Committee of Universidade Estadual do Paraná - Unespar. On April 20, 2021, the proposal was approved under opinion number: 4.661.538.

After carrying out all possible combinations between the simple proportion class –*one-to-many multiplication, division-partition, division-quotation and fourth proportional*– with the variations of the composition of measures class –*with unknown whole and with unknown part*–, we identified eight variations for the simple proportion class and composition of measures. Among them, two subclasses of problem situations were selected: simple proportion –*one-to-many multiplication*, and composition of measures– *with unknown whole*; and simple proportion –*one-to-many multiplication*, and composition of measures– *with unknown part*. The other variations were disregarded, as they did not meet the research objective.

As such, here are the didactic variables taken into account for the development of the mixed problems: *simple proportion and composition of measures; visual support; and the possibility of modelling as a function  $f(x) = ax + b$ .*

The research instrument presents four problem-situations of simple proportion problems of the *one-to-many multiplication* type with varying composition – *with unknown whole* and *with unknown part*, with two problems of the simple proportion class *multiplication – one-to-many* and composition of measures – *with unknown whole* – and two problems of the simple proportion class *multiplication – one-to-many* and composition of measures – *with unknown part*. With more advanced mathematical notation, it is possible to see that all problems can be modelled using the affine function  $f(x) = ax + b$ .

The research instrument was carefully developed following the contents and guidelines provided by the BNCC (Brasil, 2018) for the 5<sup>th</sup> grade; an analysis of the problems presented in the textbook used by the class, of the *Ápis de Matemática* collection (Dante, 2017b); assumptions of the conceptual fields theory; didactic variables; and contexts that are part of the students' reality and the specificities of a rural school.

The students, grouped into two or three, solved the four problems during class. The analyses considered the conversations between students, recorded in audio; the students' written production; and the researcher's logbook. Regarding the procedures used in data analysis, we used as support the assumptions of Vergnaud (1996b; 2009b), with regard mainly to the schemes and operative invariants expressed by the students.

In this article, we present problems 2 and 4 of the research instrument, which here we call Situation 1 and Situation 2. They were selected for their variation of visual support and due to involving in their resolutions the basic

ideas of *variable function*, *dependence*, *correspondence*, *regularity* and *generalisation*, the idea of *proportionality* and the possibility of *modelling the affine function*.

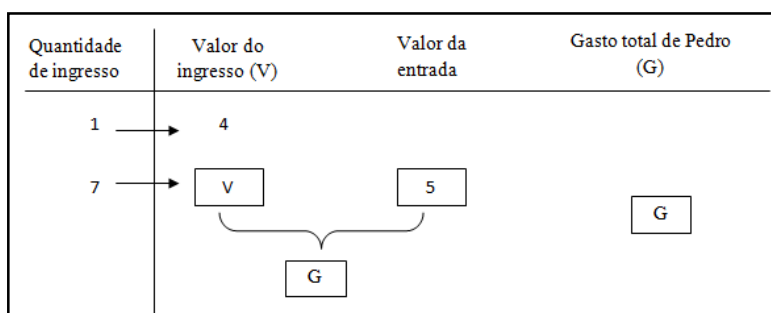
Below we present Situation 1, belonging to the simple proportion and composition of measures classes.

Situation 1: *In an amusement park, there are several rides. The ticket price for each ride costs BRL 4.00, and the entrance fee to the park costs BRL 5.00. The total amount spent at the park includes the amount paid for the tickets and the entrance fee. With this information, answer:*

- Pedro entered the park and bought 7 tickets. How much did Pedro spend in total?*
- And what if Pedro enters the park and buys 12 tickets, how much will he spend in total then?*
- How did you do the math in the previous questions?*
- After Pedro enters the park, how can we calculate how much he would spend for any number of tickets purchased?*

### Figure 6

*Sagittal scheme of the simple proportion class and composition of measures (based on Vergnaud, 2009b; Miranda, 2019; Rodrigues & Rezende, 2021)*



In this situation, the idea is that the students will find how much Pedro spent in total at the amusement park. To determine the answer to item “a,” one

must multiply the number of tickets by BRL 4.00, which represents the price of the ticket for each ride, and then add BRL 5.00 to it, which represents the value of the entrance fee, in order to find the total amount spent by Pedro. The sagittal diagram representing this situation is shown in Figure 6.

To identify “V”, we have a *one-to-many multiplication* simple proportion quaternary relation, as in Figure 7.

**Figure 7**

*Sagittal scheme of the simple proportion part (based on Vergnaud, 2009b)*

Quantidade de ingresso	Valor do ingresso (V)
1	4
7	V

We can see the relationship between the number of tickets and the amount paid for them, where one ticket corresponds to four *reais* and seven tickets correspond to V *reais*. Algebraically, this relationship can be represented as follows:  $\frac{1}{7} = \frac{4}{V}$ ,  $\logo V = 7 \times 4 = 28$ .

The relationship for the final part of the resolution, which asks for how much Pedro spent in total, corresponds to a ternary relation of the composition of measures with unknown composition, in which BRL 28.00 represents one part and BRL 5.00 the other part. Thus, the resolution of the first question results in  $V = 28 + 5 = 33$ . Therefore, in question “a”, the total amount Pedro spent when buying 7 tickets is BRL 33.00. We note that it is possible to model the resolution of this problem with the following affine function:  $f(x) = 4x + 5$ , where 4 represents the price for each ticket, x the number of tickets purchased and 5 the price of admission to the park.

To solve question “b”, the resolution process is similar to that of question “a”, changing only the number of tickets.

Question “c” was included to enable students to reflect on how they did the math for questions “a” and “b”, and to note the ideas of variable, regularity, dependence and correspondence present in this problem so that they can generalise when answering question “d”.

Next, we present the analysis of the students’ resolutions for two mixed problems, problem 1 and problem 4.

For the data analysis, we used the recorded audios, the students’ written production, and our logbook. We analysed the answers each group gave for each of the four problems. To preserve the students’ identities, in our analysis, we will refer to them as follows: student 1 and student 2 – Group 1; student 3 and student 4 – Group 2; student 5 and student 6 – Group 3; student 7 and student 8 – Group 4; student 9 and student 10 – Group 5; student 11, student 12 and student 13 – Group 6.

All problems were given to the students on an A4 size coloured bond sheet to differentiate them from routine classroom activities. Problem 1 was presented in the green sheet, problem 2 in pink, problem 3 in yellow, and problem 4 in blue.

In the first question of Situation 1, four groups presented adequate schemes for solving the question, namely: Group 1, Group 2, Group 3, and Group 6. The students in these groups expressed in their conversations and in their written protocols the understanding that they needed to multiply the number of tickets by BRL 4.00 and, at the end, add five to the result of the multiplication to find the total amount, as we can see in the fragment of the explanation given by a student in Group 1 to the researcher: “Yes... *We multiplied the ticket price by the number of ticked he bought and added five more, which was the cost to enter the park, to find how much he spent in total.*”

Figure 8 shows the resolution found by Group 1, similar to the resolution reached by the students in Group 2 and Group 3.

The resolution scheme presented by the students in Group 6, although reaching the same result as Group 1, Group 2, and Group 3, differs from the one shown in Figure 8. The students multiplied BRL 4.00 by 7 tickets, following the same pattern in all resolutions.

As an answer to question “a”, the students in Group 4 and Group 5 suggested the multiplication of the ticket price (BRL 4.00) by the number requested in the statement (7), i.e.,  $7 \times 4$ , and multiplied the park entrance fee (BRL 5.00) by the number of tickets purchased. Finally, these students added

the results obtained in these multiplications. The same reasoning was maintained when solving question “b”, changing only the number of tickets. In Figure 9, we illustrate the resolution presented by Group 5, similar to that of Group 4, referring to the first two questions of Situation 1.

**Figure 8**

Resolution of question “a” of Situation 1 – Group 1 (research files)

a) Pedro entrou no parque e comprou 7 ingressos. Qual foi o gasto total de Pedro?

$$\begin{array}{r} 7 \\ 4 \times \\ \hline 28 \end{array} \quad \begin{array}{r} 2 \\ 4 \times 28 \\ \hline 5 \\ 33 \end{array}$$

O gasto total de pedro foi 33 reais

**Figure 9**

Resolution of questions “a” and “b” of Situation 1 – Group 5

a) Pedro entrou no parque e comprou 7 ingressos. Qual foi o gasto total de Pedro?

R: 63

$$\begin{array}{r} 135 \\ +28 \\ \hline 63 \end{array} \quad \begin{array}{r} 5 \\ \times 7 \\ \hline 35 \end{array} \quad \begin{array}{r} 4 \\ \times 7 \\ \hline 28 \end{array}$$

b) E se Pedro ao entrar no parque comprar 12 ingressos, qual será o gasto total?

R: 708

$$\begin{array}{r} 12 \\ \times 5 \\ \hline 60 \end{array} \quad \begin{array}{r} 12 \\ \times 4 \\ \hline 48 \end{array} \quad \begin{array}{r} 60 \\ +48 \\ \hline 708 \end{array}$$

We observed that the students in Group 4 and Group 5 solved questions “a” and “b” of Situation 1 inconsistently with the problem. They multiplied the variable (number of tickets) both by the value corresponding to each ticket and by the price of the park entrance fee (constant); that is, in the affine function defined as  $f(x) = a \cdot x + b$ , the students presented the following false *theorem*

*in action*: If  $f$  is a functional relation, then  $f(x) = a \cdot x + b$ , with  $a, b, x \in \mathbb{N}$ .

The other groups –Group 1, Group 3, and Group 6– presented a resolution scheme for question “ $b$ ” similar to the one given in question “ $a$ ,” presenting correct solutions to the problem, except for the students in Group 2 who correctly performed the multiplication  $12 \times 4$  but forgot to add 5 to the result to find the total amount spent.

The students in Group 1, Group 3, and Group 6 got question “ $b$ ” right and, in their conversations, indicated that the resolution of this question was similar to question “ $a$ ”, changing only the number of tickets considered. Group 1 students mentioned this to the researcher while explaining question “ $b$ ”: “*It’s the same. The only thing that changed is the number of tickets.*” The students in Group 3 explained it to the researcher as follows: “*The other one we did 12 ticked, then we did  $12 \times 4$ , for the rides, and the entrance to the park, which is 5.*” The explanation given by Group 6 was: “*It’s the same thing, only the numbers changed. So here it’s 12.*”

The students in Group 6, as happened in question “ $a$ ”, reached the same result as the students of Group 1 and Group 3, but presented as a resolution scheme the multiplication of BRL 4.00 by 12 tickets.

The idea of *regularity* is not presented explicitly in Situation 1, but we expected the students to identify the existing regularity, noting that the same math performed in question “ $a$ ” is valid for other values, changing the number of tickets, since identifying regularities in a functional situation is an essential skill for developing the concept of function (Tinoco, 2002). In this sense, we observed in the dialogues and written protocols of the students of Group 1, Group 2, Group 3, Group 4, Group 5, and Group 6 that they identified the existing regularity, as they followed the same reasoning in the first two questions, changing only the number of tickets purchased, the value of each ticket, preserving the entry value.

Regarding the concept of proportionality, we observed that this idea arises through multiplication related to correspondence reasoning and through the representation of the relationship between two variables (Nunes, 2003). Thus, we verified that the students in Group 1, Group 2, Group 3, Group 4, Group 5, and Group 6 expressed this concept because, when multiplying  $7 \times 4$  or  $4.00 \times 7$  in the first question and  $12 \times 4$  or  $4.00 \times 12$  in the second one, the following proportionality is implied: 1 is to 4 as 7 is to  $x$ , in the same way, 1 is to 4 as 12 is to  $x$ , i.e., a ticket corresponds to 4.00 *reais*, and 7 tickets



correspond to  $x$  reais, and, in the same way, one ticket corresponds to 4.00 reais, and 12 tickets correspond to  $x$  reais. Algebraically, it is represented as follows:  $\frac{1}{7} = \frac{4}{x}$  e  $\frac{1}{12} = \frac{4}{x}$ . Cross-multiplying this equality, we get:  $x = 7 \times 4$  and  $x = 12 \times 4$ .

Even though in their written protocols the students in Group 6 presented the multiplications:  $4.00 \times 7$  and  $4.00 \times 12$ , their speech revealed that they understood that they needed to multiply  $7 \times 4$  and  $12 \times 4$ . However, since they used the digit four followed by two zeros after the decimal point – 4.00 – we believe they used 4.00 as the first value in the multiplication algorithm because the number of digits contained in 4.00 is greater than 7 and 12.

Analysing the resolution presented by Group 1, Group 2, Group 3, and Group 6, we can see that in the manifested scheme –bearing in mind the operations and mathematical notions expected for 5<sup>th</sup>-grade students– the ideas involved can be associated with the modelling of the function  $f(x) = 4x + 5$ , with  $(x)$  representing Pedro's total spending at the park; identified by multiplying 4, which represents the variation ratio of the function; with  $x$ , which represents the number of tickets purchased plus the constant 5 which represents the price of admission to the park.

The knowledge mobilised by the students in Group 1, Group 2, Group 3, and Group 6 can be associated with the following true theorem in action: If  $f$  is a functional relation, then  $f(x) = a \cdot x + b$ , with  $a, b, x \in \mathbb{N}$  (Rodrigues, 2021), since the students multiplied the function's variation ratio by  $x$  and the result was added to  $b$  to find  $f(x)$ .

The groups expressed the idea of *correspondence* since they correctly solved a problem situation involving simple proportions. When presenting the multiplication  $7 \times 4$  and according to the analysis of their conversations, we noticed the idea of correspondence when students stated that each ticket is equivalent to BRL 4.00. Thus, 7 tickets correspond to BRL 28.00.

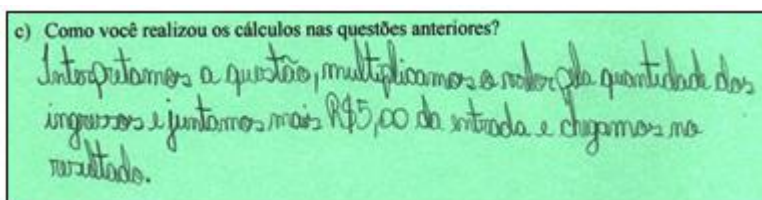
In elementary school, emphasis should not be placed on algebraic expressions or conceptual representations that express notions of *dependence*, but on relationships established between variable quantities (Brasil, 2018). In this sense, we could see that the students perceived the idea of dependence in Situation 1, even implicitly. Student 1, for example, explains: “Yes, but for him to get in, he spent five reais, right?” “He entered the park, he spent five reais. Let's do four times seven, the value plus five reais... Right?” The student

mobilises the idea of *dependence* because he knows that to find the total amount he needs to identify the amount spent on tickets and then add it to the price of the ticket, presenting the idea of dependence between two variable quantities. The resolution of Situation 1 can be modelled by the algebraic expression of the affine function  $f(x) = 4x + 5$  where  $x$  is the independent variable,  $f(x)$  is the dependent variable, 4 is the variation ratio, and 5 is the constant.

Questions “c” and “d” had to do with the basic idea of generalisation. The purpose of question “c” is to help students reach a possible generalisation of the problem. In question “c”, the students of Group 1, Group 2, Group 3, Group 4, Group 5 and Group 6 performed the operations of “multiplication and addition.” Only the students in Group 6 indicated a different answer, as seen in Figure 10.

### Figure 10

*Resolution of question “c” of Situation 1 – Group 6 (research files)*



In question “d”, the students of Group 2, Group 3, Group 4, and Group 6 presented as an answer the sum of the results they obtained in questions “a” and “b.” The students in Group 5 presented the following answer to question “d”: “addition and summing.”

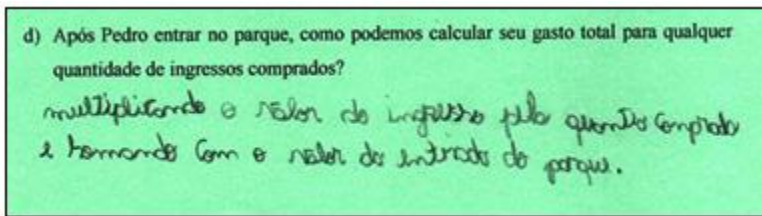
In his speech, student 9, from Group 4, explains to his colleague that in question “d”, they are not meant to sum up the results obtained in the previous questions: “No, if we had to add everything up, it would be written differently. But here it’s asking, after Pedro enters the park, how can we calculate, HOW CAN WE, it’s not asking how much he spent in total, it’s talking about how we can find it.” When talking to the researcher, student 9 says: “The ‘can’ here gives a kind of hint, ‘how can we,’ if it were the total amount, then we would have to do  $63 + 108$ .” Although the students in Group 4 were not able to

correctly answer questions “a” and “b”, they still presented a possible generalisation to the problem, even if implicitly.

The students in Group 1 were able to present a generalisation for this problem, as we can see in Figure 11.

### Figure 11

Resolution of question “d” of Situation 1 – Group 1



Student 1 was able to explain the generalisation of the problem to his colleague, however, without presenting mathematical expressions. Thus, the students in Group 1 presented the following basic ideas: *correspondence*, *dependence*, *regularity*, *variable*, and *generalisation*. The idea of *proportionality* and the possible modelling of the affine function were also mobilised. Except for *generalisation*, students in Group 2, Group 3 and Group 6 also expressed the functional ideas mentioned above.

Although the students in Group 4 and Group 5 mobilised a false *theorem in action* when solving questions “a” and “b”, we also consider that these students expressed ideas of *correspondence*, *dependence*, *regularity*, *variable*, and *proportionality*.

Situation 2: *Marcia owns a sandwich shop, and her shop's total expenditure corresponds to BRL 3.00 in ingredients for each sandwich produced and BRL 150.00 in tax per month.*

- a) *If Marcia makes 10 sandwiches, what will her total expenditure be for the month?*
- b) *Fill in the table below according to the number of sandwiches made in the month.*

<i>Number of sandwiches made by Marcia each month</i>	<i>Marcia's total monthly expenditure</i>
20	
25	
30	
35	
40	

- c) *If Marcia makes 90 sandwiches, what will be her total monthly expenditure?*
- d) *How did you do the math in the previous questions?*
- e) *How can we calculate Marcia's total monthly expenditure for any number of sandwiches she makes?*

Situation 2 brings a table as the value of the didactic variable, which differentiates it from Situation 1. The first three questions vary the number of sandwiches produced monthly so that in the last two questions, the student can generalise the calculations for any number of sandwiches produced.

To find the answer for items “a”, “b”, and “c”, one must multiply the number of sandwiches made by BRL 3.00, which represents the amount spent to make a sandwich, and then add it to BRL 150.00, which represents the tax amount. With this, we can find Marcia's total expenditure for the month.

According to the analysis of the groups' written protocols and the conversations between the students, we observed that in the first question of Situation 2, the students of three groups, Group 1, Group 2, and Group 5, managed to do the math and solve the problem.

## Figure 12

*Resolution of question “a” of Situation 2 – Group 1*

a) **Se Márcia produzir 10 sanduiches, qual será seu gasto total no mês?**

*Seu gasto total é de 180 reais*

$\begin{array}{r} 10 \\ \times 3 \\ \hline 30 \end{array}$	$\begin{array}{r} 150 \\ + 30 \\ \hline 180 \end{array}$
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We observed that the students of these three groups understood question “a” of Situation 2, as seen in the written protocol of Group 1 in Figure 12, which presents a solution scheme similar to the students of Group 2 and Group 5.

The other groups –Group 3, Group 4, and Group 6– presented resolution schemes that did not reach the answer to the problem. The students in Group 3 presented 30 sandwiches as a response. The students in this group multiplied the number of sandwiches (10) by the price of the ingredients to make a sandwich (R\$ 3.00), but they forgot to add this to the tax value.

The students in Group 4 presented a multiplication and an addition. In the multiplication, the students multiplied the amount of the tax by the price to make each sandwich, i.e.,  $150 \times 3$ . After that, the students added their results (450) to the number of sandwiches in the question (10). The answer they gave was 460.

The students of Group 6 presented as an answer to the first question the multiplication of the tax amount (150) by the number of sandwiches made that month. Their answer was 1500 *reais*.

In question “b”, the students in Group 2 filled in the table, forgetting to add the tax amount; they only did the multiplication. Although these students had solved question “a” correctly, multiplying the number of sandwiches by the production value of each one and at the end adding the tax value, in question “b”, they ended up only multiplying the number of sandwiches by BRL 3.00. Similarly, the students in Group 3 did the same procedure as the students in Group 2.

The students in Group 4 also filled in the table incorrectly. When talking to the researcher, they explained that they filled in the table by placing the digit 1 between the two digits of each number in the table.

The students of Group 6 filled in the table with the calculations that give the solution to question “b”. The students had difficulties filling in the table and, at first, they were only performing multiplication operations, forgetting to add the tax amount. When talking to them, we observed that one of the students from Group 6 asked a student from Group 1 if their solution was correct; student 1, when checking their answer, mentioned that they still needed to add the 150. There was only one mistake – when they added  $75.00 + 150.00$ , the answer they found was 227.00. We also noticed that in their resolution schemes, the students in Group 6 multiplied BRL 3.00 by the number of sandwiches made.

In the resolution presented by the students of Group 5, we observed that some values were incorrectly filled in the table. The correct total expenditure to make 25 sandwiches is BRL 225.00; for 30, BRL 240.00; and for 35, BRL 255.00. However, we understand that these students expressed an understanding of the problem and these mistakes in calculation do not compromise the mobilisation of knowledge shown by these students.

The students in Group 1 filled in the table correctly for all values, as shown in Figure 13.

**Figure 13**

*Resolution of question “b” of Situation 2 – Group 1*

b) Preencha a tabela a seguir de acordo com a quantidade de sanduíches produzidos no mês:

Quantidade de sanduíches produzidos no mês por Márcia	Gasto total no mês por Márcia
20	R\$ 210,00
25	R\$ 225,00
30	R\$ 240,00
35	R\$ 255,00
40	R\$ 270,00

$$\begin{array}{r} 20 \\ \times 3 \\ \hline 60 \end{array}$$

$$\begin{array}{r} 25 \\ \times 3 \\ \hline 75 \end{array}$$

$$\begin{array}{r} 30 \\ \times 3 \\ \hline 90 \end{array}$$

$$\begin{array}{r} 35 \\ \times 3 \\ \hline 105 \end{array}$$

$$\begin{array}{r} 40 \\ \times 3 \\ \hline 120 \end{array}$$

We noticed that the students in Group 1, Group 5 and Group 6 expressed ideas of *correspondence*, *dependence*, *regularity*, *variable*, and notions of *proportionality*. Although the students in Group 2 forgot to add the tax in question “b”, we assume that they also expressed these functional ideas, since they showed that they understood the situation when reading the statement in question “a”. When student 4 mentions: “You have to do  $10 \times 3$  and then add to 150?”, and student 3 answers: “I think so. Wait...  $3 \times 10$ ...  $30$ ...  $150 + 30$ .  $5 + 3$ ,  $8$ ...  $180$ !”, it is evident that the functional ideas were grasped.

We observed the notions of *correspondence* and *regularity* when students related that each sandwich made would cost BRL 3.00. And as the students of Group 1, Group 2, Group 5, and Group 6 presented these

calculations, we conclude that these students mobilised the concepts of *correspondence* and *regularity*.

The concept of *proportionality* is evidenced in all the multiplications of the number of sandwiches by BRL 3.00. Thus, the students of Group 1, Group 2, Group 3, Group 5, and Group 6 also expressed the notion of *proportionality*, since they showed to have understood that if a sandwich corresponds to BRL 3.00, then any number of sandwiches produced corresponds to  $x$  reais.

We consider that the notion of *dependence* was mobilised when students recognised that the total expenditure depended on the number of sandwiches made. Thus, we considered the calculations of the students who did the multiplication to find the total expenditure to make the sandwiches and then added it to the tax amount. With this, we identified the idea of *dependence* in the conversations and schemes presented by the students of Group 1, Group 5, and Group 6.

The idea of a *variable* is present in the conversations and written productions of Group 1, Group 5, and Group 6. These students followed the same calculation schemes, only changing the number of sandwiches made.

Thus, we observed that the students in Group 1, Group 5, and Group 6 mobilised functional thinking in their responses, modelled through the affine function  $f(x) = 3x + 150$ . From the answers presented in the written productions of the students in these groups, we consider that the knowledge mobilised by them can be associated with the following true *theorem in action*: If  $f$  is a functional relation, then  $f(x) = a \cdot x + b$ , with  $a, b, x \in \mathbb{N}$  (Rodrigues, 2021), because the students multiplied the variation ratio of the function by  $x$  and added the results to  $b$  to find  $f(x)$ .

For question “c”, all groups followed their table resolution scheme (question “b”) to answer the total expenditure to make 90 sandwiches. Except for the students in Group 4, who inserted the digit 1 between the numbers in the table, and in question “c”, they multiplied the tax amount by 90 sandwiches ( $150 \times 90$ ), finding an incorrect answer to the problem.

The students in Group 1 and Group 5 managed to solve question “c” correctly. They multiplied the number of sandwiches (90) by BRL 3.00, and then added the result to 150, finding the number 420.

The students in Group 6 got the same answer as Group 1 and Group 5, however, just like when filling in the table, in their resolution schemes, they

multiplied BRL 3.00 by the number of sandwiches made, that is,  $3.00 \times 90 = 270$ , and then they did  $150.00 + 270.00 = 420.00$ .

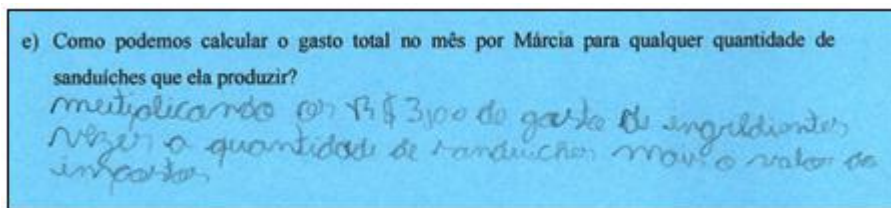
As an answer to question “c”, the students in Groups 2 and 3 presented the multiplication of 90 sandwiches by 3 *reais*, forgetting to add the result to 150 *reais*.

In question “d,” Group 1, Group 2, Group 4, Group 5, and Group 6 stated they performed multiplication and addition calculations. Group 3 multiplied  $10 \times 3$  to find the answer to question “a”.

In question “e”, we identified that Group 1 and Group 5 managed to *generalise* the problem. The other groups presented as an answer the sum of some results obtained in the previous questions. Figure 14 presents the written protocol of Group 1 for question “e” of Situation 2.

## Figure 14

*Resolution of question “e” of Situation 2 – Group 1 (research files)*



When student 1 says, “*Multiplying the three reais spent on ingredients by the number of sandwiches plus the tax*”, we see that he recognised the existing variable. That is, for any number of sandwiches, the calculations will always follow the same rule, thus presenting a *generalisation* for Situation 2. With this, we observed that the student mobilised the ideas of *dependence*, *correspondence*, *regularity*, *variable*, *generalisation*, *proportionality*, and affine function modelling.

In the written presentation given by Group 5, we noticed that the students put down “addition and multiplication” as an answer to question “e”. However, when explaining it to the researcher, it is clear that student 9 has an idea of generalisation to the problem:



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**Researcher:** *What did you understand from the last question?*

**Student 9:** *The “can” is a clue... It’s by multiplying and adding.*

**Researcher:** *But multiplying and adding what?*

**Student 9:** *Like, she made 100 sandwiches and spent 3 reais on each, so that’s 300 reais, multiplication, added to 150, that’s the addition.*

**Researcher:** *Oh, I get it. Very good, guys.*

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We noticed that student 9 referred to any number, and he cites the example of 100 sandwiches made: “*Like, she made 100 sandwiches and spent 3 reais on each, so that’s 300 reais, multiplication, added to 150, that’s the addition.*” The student is expressing all the basic ideas of function and the idea of proportionality, because he said any number (*variable*), used multiplication (*correspondence, dependence, regularity, proportionality*), and added the tax, ensuring the *generalisation* of the problem, implying the modelling of the affine function.

Thus, we identified the following function ideas in Situation 2: *correspondence, regularity, and proportionality* present in the resolutions of Groups 1, 2, 3, 5 and 6. The ideas of *dependence, variable*, and modelling of the affine function were observed in Groups 1, 5, and 6, and *generalisation* in Groups 1 and 5.

We observed that the idea of generalisation was the basic idea of function that was least identified in the groups, which is expected for the age group of students in elementary school, due to its complexity. The studies by Nogueira (2014) and Rezende, Nogueira and Calado (2020) point to this issue when showing that *generalisation* is the basic idea that causes most difficulties in understanding the concept of function.

With the analyses, we can state that the ideas of *correspondence, dependence, regularity, variable, proportionality, and modelling of affine function* were mobilised by students from all groups in at least one of the problems. The idea of *generalisation* was expressed by the students of Group 1 and Group 5.

## FINAL REMARKS

With this research, we found that elementary school students can solve situations involving notions of affine function. More specifically, the analyses show that the basic ideas of function – *correspondence, dependence, regularity, variable, and generalisation* –, the idea of *proportionality*, and *modelling of affine function* were mobilised by the students. We also observed the presence of theorems in action related to the *modelling of affine function*.

Thus, we stress the importance of analysing students' answers, which, in most cases, present implicit knowledge that can be modelled in the form of *theorems in action*. We also consider the relevance of presenting problem situations from different classes to instigate the students' development of new schemes and possible new theorems in action. In these situations, both the context involved and the didactic variables with their respective values must be considered.

We hope that this study can be a contribution to researchers interested in the subject and to elementary school teachers to make them aware of the possibility of taking the results of this study to their classrooms: the situations developed, the recognition of schemes and theorems in action that can be expressed by students, and the possibility of developing ideas of function, particularly affine functions, with elementary school students.

With the results of this research and based on Vergnaud (1996a), we argue that situations involving ideas of function and particularly of affine function can be presented to elementary school students so that during their educational process, this concept can be improved and formalised when reaching the 9<sup>th</sup> grade and high school. After all, it is from different experiences that individuals can develop new schemes, leading them to understand the concept at hand.

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