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The Construction of the Tower of Pisa as a Basis for Problem-Solving in the Exact Sciences and the Use of Interdisciplinarity in Teaching

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ABSTRACT

Background: Learning in exact sciences is a common problem for education in all countries. New education technologies are proposed to solve these situations, and using real problems in teaching is suggested. **Objectives**: This article associates Pólya's problem-solving method with the real problems in the construction of the Tower of Pisa. **Design**: In this work, we propose a multidisciplinary approach to the history of the construction of the Tower of Pisa, related to the social, historical, geological, and mainly physics and engineering problems shown in this work. **Setting and participants**: The authors were responsible for elaborating and solving the proposed problems. **Data collection and analysis**: The data used come from the history of the Leaning Tower of Pisa as a pedagogical basis for teaching, especially in the exact sciences. **Results**: Many problems and resolutions via Pólya's problem-solving method

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were performed to show teaching possibilities to teachers and students. **Conclusions**: The Tower of Pisa theme can be used to implement new education technologies, such as modelling, problem-solving, and real-life problems, as it enriches school culture and attracts an inter-and multidisciplinary character to teaching, as shown through the proposed teaching in exact sciences examples.

Keywords: Teaching; Problem-solving method; Tower of Pisa; Education, Pólya.

A construção da Torre de Pisa como base para a resolução de problemas nas ciências exatas e o uso da interdisciplinaridade no ensino

RESUMO

Contexto: A aprendizagem em ciências exatas é um problema comum para a educação em todos os países. Novas tecnologias para a educação foram e estão sendo propostas para tentar solucionar essas situações. A utilização de problemas reais para o ensino é uma das sugestões a serem utilizadas. **Objetivos**: Neste artigo, foi associado o método de resolução de problemas proposto por Pólya aos problemas reais encontrados na construção da Torre de Pisa. **Design**: Uma abordagem multidisciplinar foi proposta de acordo com a história da construção da Torre de Pisa em relação aos problemas sociais, históricos, geológicos e principalmente físicos e de engenharia que foram mostrados neste trabalho. **Ambiente e participantes**: Os autores foram os responsáveis pela elaboração e resolução dos problemas propostos. **Coleta e análise de informações**: Os dados utilizados são provenientes da história da Torre Inclinada de Pisa como base pedagógica para o ensino, especialmente nas ciências exatas. **Resultados**: Muitos problemas e sua resolução via método de resolução de problemas segundo Pólya foram realizados para mostrar as possibilidades de ensino para os professores e alunos. **Conclusões**: O tema Torre de Pisa pode ser usado para a implementação de novas tecnologias educacionais, como modelagem, resolução de problemas e uso de problemas da vida real, pois enriquece a cultura escolar e atrai um caráter interdisciplinar e multidisciplinar para o ensino, como demonstrado através do ensino proposto em exemplos de ciências exatas.

Palavras-chave: Ensino; Método de resolução de problemas; Torre de Pisa; Educação, Pólya.

INTRODUCTION

Teaching/learning in the exact sciences has been the subject of discussion around the world, both in developed and developing countries, considering the flaws and methodologies that are still linked to the old trends of solving extensive lists of exercises (Alves & Aversi-Ferreira, 2019) and a conservative type of teaching with little practical application and little relation to the activities of the future professional (Groenwald et al., 2004). Nowadays, following the ingrained conceptions of yore, teaching in different areas, with an emphasis on exact ones, is based on delivering content and then applying lists of exercises to absorb what has been taught, which has proved to be of little use for real learning (Alves & Aversi-Ferreira, 2019).

According to modern teaching technologies for problem-solving based on Pólya (1945), the proposed problems must present aspects that generate interest to the student, such as being 1) challenging, 2) interesting, 3) unknown, 4) not being a direct application of an algorithm, and 6) suitable for a certain level of difficulty (Alves & Aversi-Ferreira, 2019; Pólya, 1945). A reasonable criticism here is that modern education technologies are not always taught to prospective teachers. Those who graduated before the curricular changes do not know or understand technologies, and not all of them are constantly updating their knowledge, which generates unstructured, demotivating, and distant classes; this is the reality of the current student.

Pólya (1945) indicated a method to solve mathematical problems highlighting the following steps: 1) understand the problem, 2) establish a plan, 3) execute the plan, and 4) examine the solution. However, the extensive lists of exercises for applying algorithms are descendants of the theory of mental discipline coming from the 18th century in Germany (Groenwald et al., 2004), in which, unfortunately, monotonous repetition is still advocated as a learning method.

In this way of thinking, the student often does not know what the problems proposed in these extensive lists of exercises are about, and the resolution is made following the rules of copies of similar exercises; usually, it is a bucket of understanding, just a reproduction of what has already been seen. Learning can, of course, occur, depending on the student’s effort and the teacher’s stimulus method, but in general terms, fewer exercises focusing on the use of content reasoning would be preferable, i.e., repetitive resolution time would be used to instigate process-oriented thinking.

Although problems in teaching are also the responsibility of teachers, a recent study (Aversi-Ferreira et al., 2021) showed that most postgraduate students in the discipline of mathematical modelling prefer the old style of teaching, i.e., asking the teacher for lists of exercises similar to given examples instead of looking for ways to solve proposed problems based on real data. What happened was a proposal within the practical data of an exercise based on real facts, and the student would need to look for means within the content to solve. However, the proposal failed when students asked for a list of similar exercises they could use to solve the proposed exercise. Therefore, the responsibility for problems in the insertion of new technologies in education must be shared with students.

This is a complex discussion that deserves to be studied elsewhere, as students already used to a method during their education will find it difficult to accept another, especially one that requires reasoning and directed intellectual work.

An adequate strategy for teaching is to correlate the studied content with applications and examples already known from the student's reality. This facilitates interdisciplinarity, as most examples based on reality involve several areas for resolution, while exercises from a list are overly directed to only a few topics of the disciplines, minimising the process of relationship with reality (Alves & Aversi-Ferreira, 2019).

Alternatively, facts with examples that include a mnemonic search already widely known by the population, by deduction, can facilitate learning, as it can generate interactions between students with colleagues, parents, and social circles and make the subject interesting and motivating. Within this scope, the Leaning Tower of Pisa is valuable, as it provides data on engineering, mathematics, and physics, in addition to soil analysis, history, and sociology.

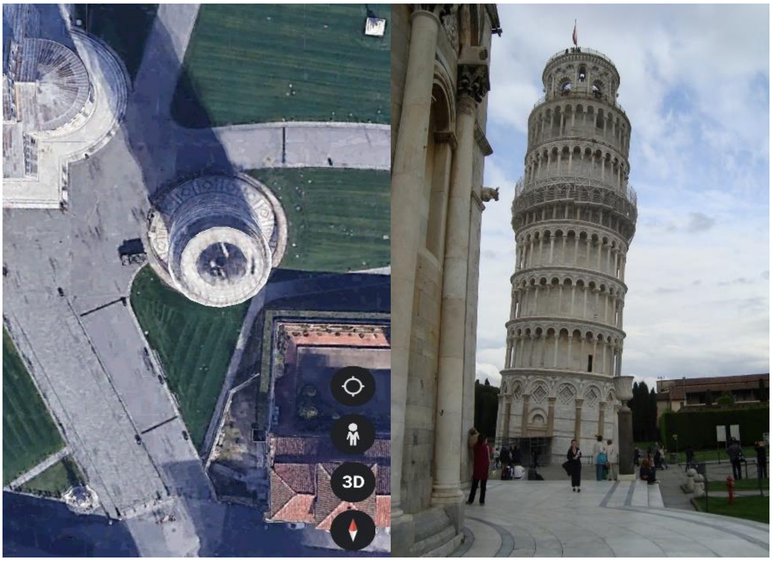
Knowing the history of the Tower of Pisa and the philosophy of the long period of its construction, this work has as its main objective the analysis of the countless physical properties that can be explored through an interdisciplinary study of history, philosophy, mathematics, physics, and engineering. With its vast applications, science allows us to explore different events, and the historic landmark of the Tower of Pisa that leaned centuries ago is one example. The present study takes advantage of concepts such as the Tower of Pisa's force distributions, the rotation of coordinate systems, Galileo's experiment, the scientific method, volume, the centre of mass, the critical angle calculation, and trigonometric ratios offered by the Tower case, which can be applied to teaching the exact sciences, especially for civil engineering students.

THE TOWER OF PISA

The Leaning Tower of Pisa (Figure 1), which was supposed to represent the wealth and prosperity of the city of Pisa, an independent trading port, took almost 200 years (1172 to 1370) to build (Burland, 1998; Burland et al., 2009). Its construction was in three phases (Bajaj & Choudhary, 2014) as an independent appendix to the cathedral of Pisa. Interestingly, the Leaning Tower of Pisa has become one of the most historical symbols of architecture and engineering worldwide due to its famous incline (Bajaj & Choudhary, 2014). It is the smallest tower in height to be considered one of the greatest engineering works due to its construction that employed medieval engineering technology.

Figure 1

Tower of Pisa. A) Upper view (Google Earth). *B) Frontal view* (personal archive, 2010)



History mentions that in 1172, the widow Berta di Bernardo bequeathed 60 gold coins in her will for the purchase of stones for the construction of the Bell Tower, which was behind the Cathedral of Pisa. A year later, in August 1173, the construction began (Burland, 1998) under the supervision of Bonano Pisano. The third oldest in the city, the tower was to be independent of the cathedral and baptistery (Bajaj & Choudhary, 2014), circular and with elaborately carved columns with complexes in bas-relief.

The tower was raised as a masonry cylinder with six columns, a typical work of medieval engineering (Bajaj & Choudhary, 2014). After five years of construction, in 1178, and with only four floors built, the foundations were found to be shallow and inadequate. The subsoil was not very solid and unstable, which made the tower lean (Bajaj & Choudhary, 2014; Bartelletti et al., 1988). The construction stopped for 100 years for the first time, according to some authors, for financial reasons due to the war between Pisa and Florence. According to other researchers, the cause of the stoppage is unknown (Burland, 1998). At that time, the ground had settled (Bajaj & Choudhary, 2014).

In 1275, more floors were added, under the supervision of Giovanni di Simone (an architect from Camposanto), who added more height to the floors opposite the slope, during six years of work (Bartelletti et al., 1988). Today, the Tower of Pisa is curved, with the top sloping away from the initial sloping side (Bajaj & Choudhary, 2014).

Construction came to a halt again due to conflict between Meloria, Pisa, and Genova (Burland, 1998). The seventh floor was completed in 1319, with work carried out by the architect Tommaso di Andrea Pisano, who harmonised the Gothic elements with the Romanesque style of the tower. The bell tower was finally started in 1360, and it was finished in 1370 (Bartelletti et al., 1988; Burland, 1998). Disregarding the leaning of the Tower of Pisa, its construction demonstrates the clever use of arches and columns to support and distribute weights and loads, something beyond expected from a medieval construction. The first two floors have 15 arches each, made with closed marble, while the other floors contain 30 arches and the bell tower has 16. However, the builder did not consider or realise the clayey soil that would need to support a weight of 16 million kilograms. The identity of this architect is controversial; some believe that it was Bonano Pisano, a well-known artist from the 12th century who lived in Pisa, but others believe that it was Diotisalvi, the author of the baptistry project (Bartelletti et al., 1988).

The structure of the tower is cylindrical, made of limestone and lime mortar (several openings were found in this mixture), with the outside clad in marble (Bartelletti et al., 1988; Burland, 1998; Burland et al., 2009). Due to its flexibility and ability to withstand stresses well, limestone is the structure responsible for the lack of cracks in the structure of the tower, and the reason for its not falling over.

The Tower of Pisa has eight floors, with 55.88 m on the lowest side and 56.67 m on the highest side, a base of 15.484 m in diameter, located 5.5 m below ground level, and has 296 steps on the south side and 294 on the north side. Seven bells were placed, one for each musical note (Bajaj & Choudhary, 2014).

There have been several attempts to decrease the tower's inclination over time. Initially, in 1272, engineers tried to compensate for the tilt by making the upper floors on one side higher than the other; however, this added to the weight, and the tower sank and tilted further. In 1838, the architect Alessandro Della Gherardesca excavated the tower to expose the foundation; however, as the opening was below the water level, the slope increased by 0.5° due to infiltration (Burland, 1998; Johnston & Burland, 2000).

In 1911, systematic measurements of the movement of the tower began, showing an inclination of around 5° from north to south (Bartelletti et al., 1988). The southward slope is believed to have started with the installation of the bells (Burland, 1998). In 1934, 361 perforations were made in the base and filled with 99 tons of cement, improving the structure (Bartelletti et al., 1988; Burland, 1998) but, as a consequence, the tower moved 10 mm to the south.

In the 1960s, the Italian Ministry of Public Works created an international commission to study the structure and mechanics of the ground under the Tower of Pisa, and, in 1971, delivered technical documentation suggesting an international competition for the stabilisation of the structure (Bartelletti et al., 1988).

In 1984, Raffaelo Bartelleti, Glorglo Berardi, and Luciano Caroti formed a team appointed by the Ministry to stabilise the tower. The experts proposed two steps toward the solution. The first was stabilising the moment of force, and the second involved completely shoring up the existing foundation. The second would occur depending on the outcome of the first step (Bartelletti et al., 1988).

The implementation of the first step involved the temporary use of steel structures placed on a circular concrete beam located far enough from the tower and then building structures under the it in mini piles with high bearing capacity after removing part of the subsoil under the tower; according to Burland (1998), this increased the slope by about 10 arcseconds, approximately 0.003º.

During the last hundred years of construction, there were several attempts to correct the structure of the tower, but to no avail. It was not until 2008 that engineers verified that the tower had stopped leaning since it was built.

THE HISTORY OF THE TOWER AFTER MODERN ENGINEERING

In the 1990s, the Tower of Pisa was closed to the public. In March 1990, the Prime Minister of Italy appointed a new multidisciplinary commission, including structural and geotechnical engineering, architecture, architectural history, archaeology, and restoration, under the direction of Professor Michele Jamiolkowski, to take steps to stabilise the tower (Burland, 1994; Burland et al., 2009).

The great challenge of modern engineering was to stabilise a tower founded on fragile and sandy soil, whose support structure will also be supported on that same soil. This challenge spurred the authors to study this great engineering work.

This commission could, for the first time, use computer models to predict the behaviour of the tower in the face of stabilisation hypotheses (Burland, 1998). The computer models would need reliable information about the tower's inclination through more exact measurements. One fact became clear: previous attempts to correct the slope had increased it!

The computational analysis used a finite geotechnical element program known as ICEFP, developed by Imperial College (Potts & Gens, 1984), based on the critical stage concept (Schofield & Wroth, 1968) and non-linear elastic-plastic hardening, to try to understand the behaviour of the tower's tilt (Burland, 1998; Burland & Potts, 1995).

The Tower of Pisa was reconstructed step by step via a computational model, and a good relationship between the model and reality occurred, mainly about the slope value (Burland, 1998).

Burland and Potts (1995) concluded, via computational analysis, that the Tower's instability is due to the phenomenon called 'leaning instability' by Edmund Hambly (1985), which is not due to a lack of soil support capacity, but to insufficient rigidity, due to the so-called 'Pancone clay' or soft clay (Burland, 1998).

The subsoil under the tower is made up of three main layers. The first, about 10 m thick, is made up of sandy sediments and soft clay. The second is formed by clay of marine origin, and is very sensitive, with about 40 m of depth. The third layer is formed by dense sand, of marine origin, about 60 m deep (Burland et al., 2009). To stabilise the tower, two problems would need to be solved: stabilising the masonry structure that suffers from high structural stress due to the slope and stabilising the foundation (Burland, 1998).

For the first stage, lightweight prestressed plastic cables covered with steel were attached to the first-floor and second-floor spans (Burland, 1998). The second stage was carried out with the placement of concrete weights in the form of rings on the north side of the tower to stabilise it on the soft clay layer. Placement of the rings began in May 1993 and was completed in January 1994. By February 1994, the tower had decreased its slope by 2.5 mm (Burland et al., 1994).

After placing the rings, excavation took place under the foundation on the south side and the placement of a water table on the north side, and with the decrease in inclination, the Tower of Pisa was then reopened to the public in 2001 (Burland, 2002; Burland et al., 2009). In two decades, and at an expense of 25 million dollars, the work of this last commission left the tower with an inclination of 3.97º or 3.9 mm; this was about 5.5º at the beginning of the work.

The data commented on in the form of history presents a rich source for educational studies in various areas of knowledge, both in secondary and higher education. Within this, the objective of this work is to provide exercises and debates with questions and provide answers to examples of problem-solving processes within the scope of new educational problem-solving technologies (Alves & Aversi-Ferreira, 2019; Pólya, 1945) in the exact sciences and in an interdisciplinary manner with geography, history, and sociology.

INTERDISCIPLINARITY IN THE STUDY OF THE LEANING TOWER OF PISA IN GENERAL TERMS

With the appeal of new didactic technologies for quality teaching, interdisciplinarity can be explored by using the Leaning Tower of Pisa as an example for exercises in different areas. Sociology is an area where the Tower of Pisa can be widely discussed, as the Tower of Pisa was started with money from a woman, the widow Berta di Bernardo, who, in 1172, left money to build a bell tower.

The first proposal encompasses a fruitful discussion about women's societal role over time. For example, questions can centre on 1) how women were treated in the 12th century and 2) whether the widow, because she was rich, was treated differently than other less wealthy women without a husband. For example, did the ancient law grant monetary control of the husband's possessions to all widows? Do rich women also suffer less discrimination today? These are some questions to be asked in a sociological and historical discussion using the Tower of Pisa.

Historically, from 1173 to the present day, the construction of the Leaning Tower of Pisa has gone through several events, such as the work being interrupted in times of war due to a lack of money. Historical and sociological discussions about the consequences of war, such as the lack of money for social work and food for the people, can be used as a reference to show the harm and strange motives of many wars and the consequences for the population. This issue has been widely discussed, as the war between Russia and Ukraine is generating economic changes worldwide (Ali et al., 2022). This is the second proposal.

In a multidisciplinary teaching process, the exact sciences should be considered/prioritised for students who have difficulties in mathematics and physics. Many of those students may show an interest in the humanities; therefore, the history of the Tower of Pisa may raise such interest, reducing the fear of exact disciplines during teaching. In this way, the students realise that mathematics and physics are part of everyday life and part of history. These proposals will serve, according to the teacher's needs, to both high school and higher education students.

The other proposals using the Leaning Tower of Pisa focused on the exact sciences are detailed below.

INTERDISCIPLINARITY IN THE EXACT SCIENCES

Notwithstanding the analysis of movement and forces carried out in Ancient Greece, mainly by Aristotle in experimental terms, and considering the thrust studied by Archimedes as an experimental work, practical analysis began in the modern age with Francis Bacon and René Descartes at the beginning of the 17th century with the advent of the scientific method (Voit, 2019). Also in the 17th century, Galileo Galilei and Isaac Newton performed mathematical analyses associated with experiments in the studies of physics and motion. Newton made this analysis more complete, focusing on the method of flows applied to motion using the ideas of differential calculus he discovered concomitantly with Leibnitz (Nogueira, 2016).

METHODOLOGY

A bibliographic survey was carried out by searching for data on the Leaning Tower of Pisa to qualify the proposals of the exercises within the CAPES [Coordination of People Improvement of Superior Level – Brazil] periodicals platform that allows for finding texts within other databases such as Scielo, Scopus, and Web of Science. Additional material was supplied by searching within Google Academic. The main works consulted dealt more specifically with the construction of the Leaning Tower of Pisa with information on the scientific method, engineering, physics, teaching, and others. In total, 23 texts were broken down and separated by subject (Table 1, Figure 2).

Table 1

Works that specifically deal with the construction of the Leaning Tower of Pisa

Texts	Subjects	Type
1. Ali et al., (2022). The Economic Implications of the War in Ukraine for Africa and Morocco.	Other	Article
2. Alves & Aversi-Ferreira (2019). Comments on the problems solving methodology in education of civil engineering in Brazil.	Teaching	Article
3. Aversi-Ferreira et al., (2021). The perceptions of students and instructor in a graduate mathematical modeling class: An experience with remote education.	Teaching	Article
4. Bajaj & Choudhary (2014). Outstanding Structure: The Leaning Tower Of Pisa.	Engineering	Article
5. Bartelletti, Berardi & Caroti, (1988). Stabilisation of the Leaning Tower of Pisa	Engineering	Article
6. Burland (1998). The enigma of the leaning of the tower of Pisa.	Engineering	Article
7. Burland (2002). The Stabilisation of the Leaning Tower of Pisa.	Engineering	Article
8. Burland, Jamiolkowski & Viggiani (2009). Leaning Tower of Pisa: Behaviour after Stabilization Operations.	Engineering	Article

9. Burland et al., (1994). Pisa updatebehaviour during counterweight application.	Engineering	Article
10. Burland & Potts (1995). Development and application of a numerical model for the leaning tower of Pisa.	Engineering	Article
11. Crombie (1957). ***Augustine to Galileo: The History of Science A.D. 400-1650.***	Physics	Book
12. Frizzarini, & Cargnin (2005). ***Prática de Ensino: Novas tecnologias e jogos didáticos.***	Teaching	Book
13. Groenwald, Silva & Mora (2004). ***Perspectives in Mathematics Education.***	Teaching	Article
14. Halliday & Walker (2013). ***Fundamentals of Physics*****.**	Physics	Book
15. Hambly (1985). Soil buckling and leaning instability of tall structures.	Engineering	Article
16. Hibbeler (2016). Engineering mechanics. Statics.	Physics	Book
17. Johnston & Burland (2000). An Early Example of the Use of under excavation to stabilise the Tower of Stchad, Wybunbury in 1832.	Engineering	Article
18. Newburgh & Andes (1995). Galileo Redux or, how do nonrigid, extended bodies fall?	Physics	Article
19. Nogueira (2016). História da matemática.	Teaching	Book
20. Pires (2011). A evolução das ideias da Física.	Physics	Book
21. Pólya (1945). How to Solve It: A New Aspect of Mathematical Method.	Teaching	Book
22. Potts & Gens (1984). The effect of the plastic potential in boundary value problems involving plane strain deformation.	Physics	Book
23. Schofield & Wroth (1968). Critical state soil mechanics.	Engineering	Article
24. Voit (2019). Perspective: Dimensions of the scientific method.	Scientific method	Article

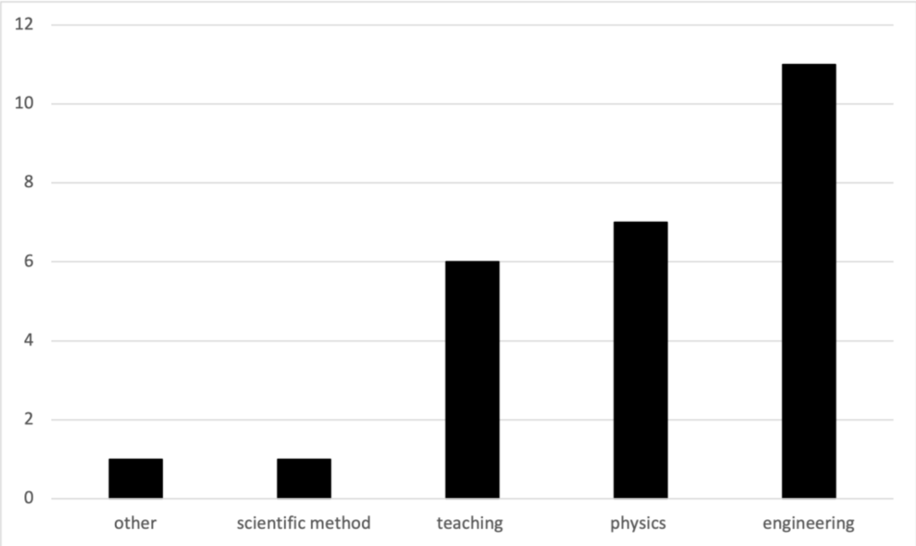
25. Volkwyn et al., (2020). Learning to use Cartesian coordinate systems to solve physics problems: The case of 'movability.'	Physics	Article

The sources and their respective subjects are shown in Table 1, considering the interdisciplinarity carried out when working on the Leaning Tower of Pisa. Later, a review covered the different areas of study.

The texts presented problems that the authors considered to have the potential for teaching in the exact sciences, looking for points of contact with other areas to foster an interdisciplinary process.

Figure 2

Comparative graph of the subjects of the texts studied to generate the epistemological bases of this work.



RESULTS AND DISCUSSION

Twenty-five texts were consulted for the preparation of this article. The texts were used as a basis for the preparation of the problem proposals with the Leaning Tower of Pisa as a substrate. Specifically, about the Leaning Tower of

Pisa, most of the articles dealt with engineering and history. Of the other texts, such as those on physics, one was on the history of Galileo and the experiment on falling bodies, while the others were on physics, engineering, and education and used for the theoretical foundation of the problems involving calculus.

Case study: forces

The four steps for solving problems indicated by Pólya (1945) can be used in the reasoning process, and we shall comment concomitantly in teaching the proposed problems. Pólya's steps proposed are:

i. Read and understand the problem.
ii. Establish an action plan.
iii. Execute the plan.
iv. Examine the solution.

So, according to the interpretation of the Pólya's method, we must verify the data, the incognita, the conditions and/or the restrictions and whether the conditions are enough to determine the incognita in **reading and understanding the problem**.

Then, once space and time are defined, the referential system of an object is determined (Halliday et al., 2013). To study the tower, the reference is inertial and can be the Earth or any object in the surroundings; it is better to choose the Earth. For this proposal, which is to define the inertial reference, the Tower of Pisa is stationary, and we can verify that the resultant force ($\vec{F}_R$) that acts on it is equal to zero (Hibbeler, 2016):

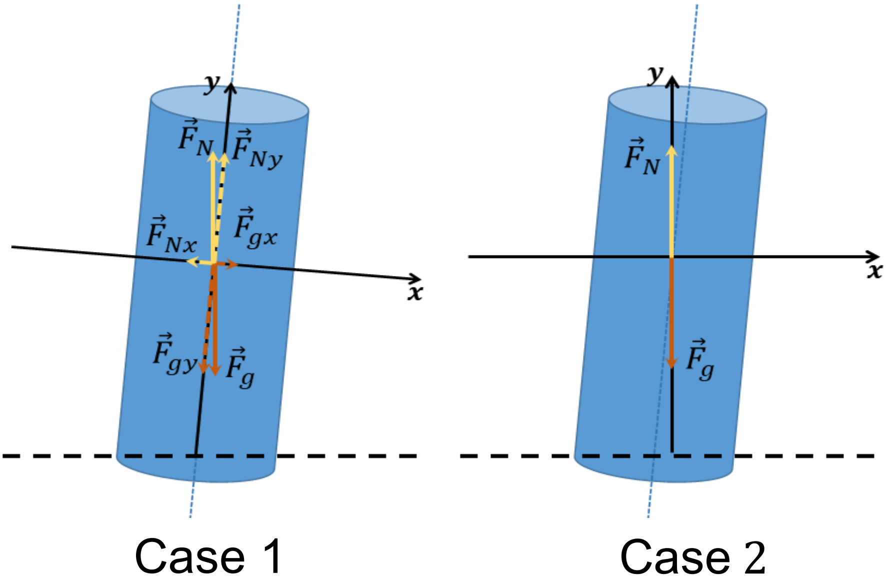
$$\vec{F}_R = \sum_{i=1}^{n} \vec{F}_i = m\vec{a} = 0 \, .$$

In this case, after searching the necessary data, the incognita, and the conditions, the conclusion of the first step was that $\vec{F}_R = 0$.

Figure 3 and its reasoning are made to **establish an action plan**, according to Pólya (1945). Indeed, the study of theory in the books about the statics is to search for a similar problem as the necessary formulas to help the solution.

Figure 3

Reference systems. Case 1: y axis parallel to the vertical axis of the Tower of Pisa. Case 2: x axis parallel to the ground.



Then, the reasoning for the steps is as follows. For the basic teaching of static systems, two types of analysis can be proposed: case 1, the Cartesian system with the y-axis parallel to the inclined vertical axis of the tower, and case 2, with the x-axis parallel to the floor (Figure 3).

Naturally, in case 1, the x-axis component of the gravitational force ($\vec{F}_{gx}$) will have an inclination angle of 4º concerning the horizontal plane of the Earth's surface, and, in case 2, the x component is zero. The student will be able to see that two axes are enough and that a two-dimensional analysis will provide data on the stability of the tower.

In continuation, we will see that the Cartesian system can be suitable for the studied system and calculate the components of forces with the data provided from the tower, such as the angle of inclination and its mass (m), approximately $14.7 \times 10^6\ kg$.

Using vector notation, we find that the gravitational force acting on the tower is:

$$\vec{F}_g = \vec{F}_{gx} + \vec{F}_{gy}$$
$$\vec{F}_{gx} = \vec{F}_g \, cos(\theta) \qquad (1).$$
$$\vec{F}_{gx} = \vec{F}_g \, sin(\theta)$$

In scalar notation, these components can then be calculated as $F_{gx} = F_g \, cos(\theta)$ and $F_{gx} = F_g \, sin(\theta)$ being $F_g = mg$ and $g = 9.81 \, m/s^2$. In unit vector notation, this is $\vec{F}_g = F_{gx}\hat{\imath} + F_{gy}\hat{\jmath}$. In both cases, vector notions can be established and related to scalar notation.

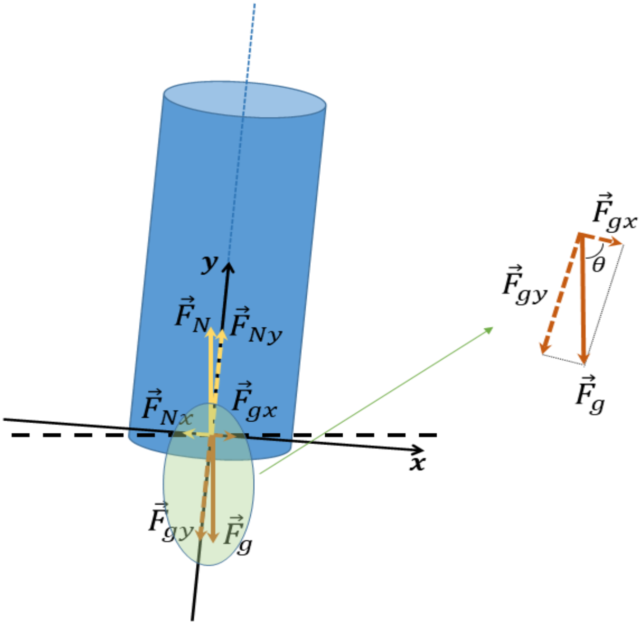
Calculating the gravitational force components with the angle of inclination to the y-axis can be done using the midpoint of the transverse line segment of the tower's base. As the Tower of Pisa is an extensive body and the weight[2] supported by the foundation is what matters, a relationship between theory and practice can be discussed.

Ideally, all the weight of the tower is placed at a point, i.e., the midpoint of its base (Figure 4), mentioned above, and the components of the gravitational force are calculated for that point. It is shown to the student that it is enough to make the centre of mass of the tower coincide with the centre of coordinates and that the position of the latter will not matter since the distance does not participate in the calculation given the angles. Thus, the mass and the acceleration of gravity (g) can be considered the same for the entire length of the tower.

The third step proposed by Pólya is the **execution of the plan**. Then, we apply the values in the formulas of the components studied above, using quantities with the units in the International System of Units (kg for mass and N for forces). For high school, the value of the cosine of 86º can be provided, or, according to new ideas in mathematics education, the use of a calculator can be introduced for checking accounts, correcting errors, and visualising results (Frizzarini & Cargnin, 2005). The plan execution to solve the force problems is shown below, and it is important to verify whether each step is correct and whether it is possible to obtain proof for each step.

Figure 4

Representation of forces applied to the Tower of Pisa: gravitational force $(\vec{F}_g)$ *and normal force* $(\vec{F}_N)$.



The formulas used are:

$$F_{gx} = F_g \cos\theta$$
$$F_{gx} = mg \cos\theta$$
$$F_{gy} = F_g \sin\theta$$
$$F_{gy} = mg \sin\theta$$

The data were placed on the formulas:

$$F_{gx} = F_g\, cos\, \theta$$
$$F_{gx} = mg\, cos\, \theta = 14.7 \times 10^6\ kg \cdot \frac{9.81m}{s^2} \cdot cos\, 86^o = 1.006 \times 10^7 N$$
$$F_{gy} = F_g\, sin\, \theta$$
$$F_{gy} = mg\, sin\, \theta = 14.7 \times 10^6\ kg \cdot \frac{9.81m}{s^2} \cdot sin\, 86^o = 1.439 \times 10^8 N$$

The solution examination, in this case, is to verify whether the data were placed correctly and calculate again to verify whether the calculus is correct. For that, verification of real data is a kind of reference in the final calculation.

After checking the results, attention should be drawn to the difference in values between the x and y components and, especially, the existence of the x component, which should ideally not exist in a construction of this type.

The problem follows to another stage, and Pólya's steps were used but not indicated as in the above example. The search for theories and/or analogous problems were performed using the references and analysis of the situation, showing that the problems were not solved directly but carefully, in function of the theory behind each proposed problem.

Then the result, due to the x component being different from zero, can be calculated by the Pythagorean theorem:

$$F_g = \sqrt{{F_{gx}}^2 + {F_{gy}}^2}$$
$$F_g = \sqrt{(3.15 \times 10^7 N)^2 + (1.41 \times 10^8 N)^2} = 1.442 \times 10^8 N$$

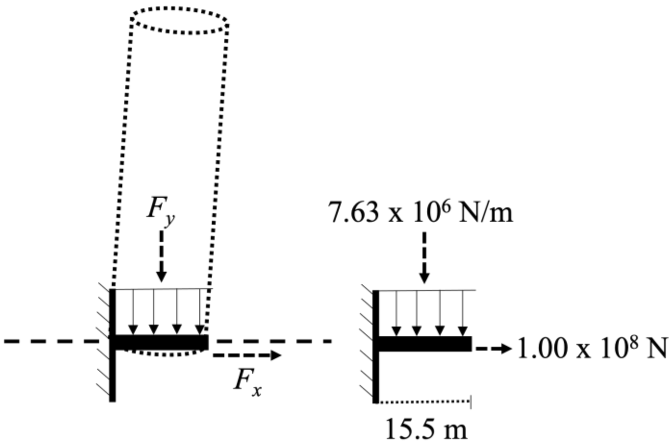
This calculation, associated with Figure 4, will show the imbalance of the tower with the resultant facing the fourth quadrant of the Cartesian coordinate system.

The calculation was performed idealised for a point of concentrated mass, but calculations can be made considering the tower as an extensive body to teach in higher courses.

The notion of normal can be shown and recalled in this problem, showing that it originates from the contact of surfaces and is perpendicular to the surface, with the same direction of action of the gravitational force, but in the opposite direction (Halliday et al., 2013).

Figure 5

Idealised beam to support the tower



For an engineering course, it is clear that there is a moment and a shear or shear force in the tower because the x component is different from zero. An idealised beam can be drawn at the base of the tower to support its weight, and the reactions can be calculated. The beam idealised for a tower whose y-axis was perpendicular to the transverse axis on the ground would have full and continuous support, but the angle of 4º indicates that a part of the side works as a cantilever beam due to the obtuse angle of the tower about the y-axis. Because this idealised beam provides support, in addition to the weight of the tower, it is also subject to the reaction of the component $\vec{F}_x$ $(\vec{F}_{gx})$.

New data must be added for the resolution, such as the base diameter (d) of 15.5 m. It is obvious that the beam has area and volume, but the calculations are made considering a two-dimensional analysis (Figure 5). To verify the force that the wall on the obtuse angle side supports, we idealised a beam fixed to this wall supporting the weight of the tower as a distributed force and the reaction to the component $\vec{F}_y$ $(\vec{F}_{gy})$.

The beam is isostatic with three unknowns and three reactions (the components of the gravitational force and the moment). The distributed force (F_d) is calculated by the component F_y divided by the area of the base of the tower:

$$F_d = \frac{F_y}{\pi \cdot (\frac{d}{2})^2} = \frac{1.439 \times 10^8}{188.692} \frac{N}{m^2} = 7.624 \times 10^5 \frac{N}{m^2} \quad (2).$$

The calculations for the beam follow the principle of rigid body equilibrium (Hibbeler, 2016), in which the sum of the resultant force and the moment are equal to zero, and can be calculated directly by analysing the beam or using an integral calculation (Figure 6):

$$\vec{F}_R = \sum_{i=1}^{n} \vec{F}_i = 0$$
$$\vec{M} = \sum_{i=1}^{n} \vec{M}_i = 0 \quad (3).$$

For the reaction to the distributed force (Ra), we have

$$\text{How } \sum_{i=1}^{n} \vec{F}_{yi} = 0 \;\Rightarrow\; R_a - F_d \cdot d = 0$$
$$R_a - 7.624 \times 10^5 \frac{N}{m^2} \cdot 15.5m = 0 \;\Rightarrow\; R_a = 1.182 \times 10^7 \frac{N}{m} \quad (4).$$

For the reaction to the force F_x (N_e), we have

$$\text{How } \sum_{i=1}^{n} \vec{F}_{xi} = 0 \;\Rightarrow\; N_e - F_x = 0$$
$$N_e - 1.006 \times 10^7 N = 0 \;\Rightarrow\; N_e = 1.006 \times 10^7 N \quad (5).$$

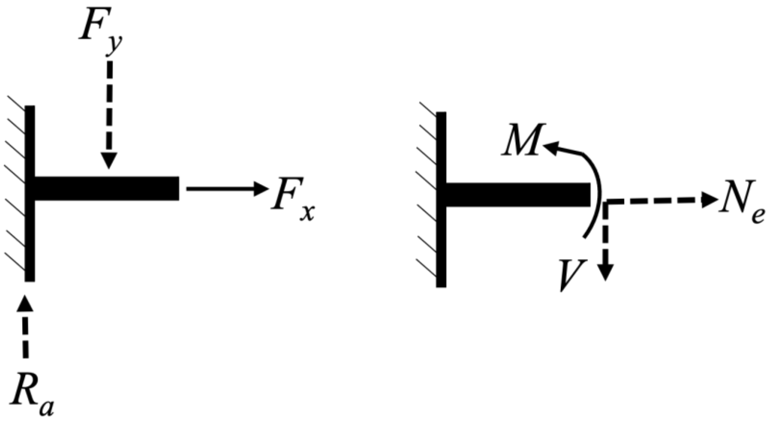
For the moment (M), we have

$$\text{How } \sum_{i=1}^{n} \vec{M}_i = 0 \;\Rightarrow\; M + R_a \cdot \frac{d}{2} = 0$$
$$M + 9.158 \times 10^7 N = 0 \;\Rightarrow\; M = -9.158 \times 10^7 N \quad (6).$$

The equations for the shear or shear force (V) and the moment from it are then calculated (Figure 6).

Figure 6

Variables involved in the analysis of a beam indicated in the scheme.



The shear force V can be calculated as

$$-V(x) - F_d \cdot x + R_a = 0 \Rightarrow V(x) = R_a - F_d \cdot x$$
$$V(x) = 1.182 \times 10^7 \frac{N}{m} - 7.624 \times 10^5 \frac{N}{m^2} \cdot x \quad (7).$$

The moment M can be calculated by analysis of the structure (see equation 9) or from the integral of the shear force; this is the teacher's choice.

$$M = \int V(x)\ dx$$
$$M = \int (R_a - F_d \cdot x) dx = R_a \cdot x - F_d \cdot \frac{x^2}{2} + C \quad (8).$$
$$M = 1.182 \times 10^7 \frac{N}{m} \cdot x - 3.812 \times 10^5 \frac{N}{m^2} \cdot x^2 + C$$

To calculate the constant C, it is enough to obtain the value of C for x equal to 15.5 m, which is the length of the beam and with M=0, then we have

$$C = 7.624 \times 10^5 \frac{N}{m^2} \cdot (15.5m)^2 - 1.182 \times 10^7 \frac{N}{m} \cdot (15.5m) = -9.158 \times 10^7 N$$
$$M = -3.812 \times 10^5 \frac{N}{m^2} \cdot x^2 + 1.182 \times 10^7 \frac{N}{m} \cdot x - 9.158 \times 10^7 N$$

(9).

If one wants to find the direct value of the moment, one applies the definite integral over the length d, the diameter of the base of the tower

$$M = \int_0^d V(x)\, dx \qquad (10).$$

Other calculations can be performed from the one shown for the exact sciences, such as the tower's centre of gravity or the maximum displacement as examples.

Case study: rotation of coordinate systems

We demonstrated above that to study the Leaning Tower, we can change the Cartesian coordinate system, and this rotation of references defines that the coordinates of the position vector in the two systems are related in such a way that the position vector (or in any case another vector) is kept invariant.

This change concerns a geometric aspect of the problem. From a mathematical point of view, a rotation is like a linear transformation involving coordinates; in physical terms, all the physical properties of the body are maintained. Changing coordinates can facilitate solving some problems (Volkwyn et al., 2020).

As proposed for the upper level, let any vector A pass through the inclined axis of the tower, forming a line segment joining the midpoints of the base and dome of the tower, in the base-dome direction for vector A (Figure 7).

By definition, θ is the angle resulting from (θ' + φ). Considering that Ay = A · cosθ and $A_z = A \cdot sen\theta$

$$A_y = A \cdot cos\theta' = A \cdot cos(\theta - \varphi) \qquad (11).$$

We will use the expression for the cosine of the difference of the arcs φ

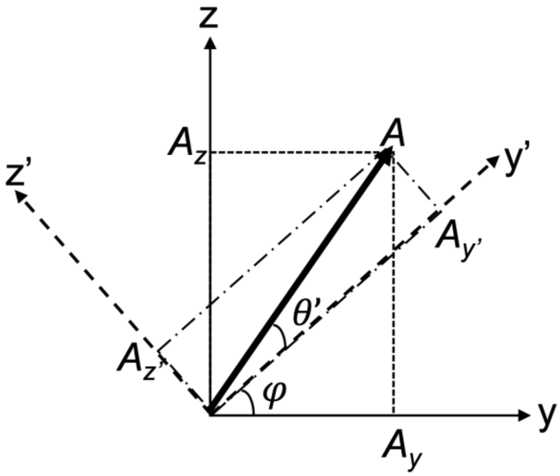
$$A_y = A \cdot cos\theta \cdot cos\varphi + A \cdot sen\theta \cdot sen\varphi.$$

We are describing A_y(y') as a component of vector A in the rotated system, in terms of the components of vector A in the original system

$$A_y = A_y \cdot cos\varphi + A_z \cdot sen\varphi.$$

Figure 7

Representation of coordinates i and i'.



Retracing for the z axis

$$A_z = A \cdot sen\theta' = A \cdot sen(\theta - \varphi).$$

So, we expand the sine of the arc difference

$$A_z = A \cdot sen\theta \cdot cos\varphi - A \cdot cos\theta \cdot sen\varphi,$$
$$A_z = A_z \cdot cos\varphi - A_y \cdot sen\varphi.$$

The components of vector A were written in the run system in terms of the original components. To have a condensed representation of this transformation, we will use the following matrices to represent the planes:

$$A = \left(A_y\ A_z\ \right),$$
$$A = \left(A_{y'}\ A_{z'}\ \right),$$
$$\left(A_{y'}\ A_{z'}\ \right) = (cos\theta \cdot sen\varphi\ \ - sen\varphi \cdot cos\varphi\ ) \cdot \left(A_y\ A_z\ \right).$$

The determinant of the rotation matrix is given a value of 1, which means that the transformation does not change the norm of the vector. When the vector is transformed by rotating it, this transformation is maintained, and its modulus is preserved

$$A_i = \sum_{j=1}^{2} R_{ij} \cdot A_j.$$

The above is a condensed way of demonstrating this transformation, known as a rotation matrix. Einstein's notation tells us that we can omit the summation, so

$$A_i = R_{ij} \cdot A_j.$$

Taking the Leaning Tower of Pisa as a reference, this is an action proposal for physics teaching with the calculation of the gravitational force and its components. Furthermore, this exemplifies the use of matrices and change of basis for studying analytical geometry with linear algebra aimed at higher education students and the transfer of an inertia centre in civil engineering studies.

Case study: galileo's mythical or real experiment and the scientific method

Although several high school books, such as Paul Hewitt's (Conceptual Physics), claim that this experiment was factual, other authors state its origin is doubtful regarding the facts (Pires, 2011; Crombie, 1957). Nevertheless, it is considered the second most beautiful experiment in the history of physics by *Physics World* magazine. Galileo's example generates an intersection with the discipline of history and a discussion about the veracity of facts and the reason for different interpretations of certain subjects.

In a hypothetical experiment, Galileo wanted to refute Aristotle, who claimed that bodies with different masses fall at different speeds. Galileo already knew the answer to this experiment, but he decided to prove it. Under the conditions of the time, he used the Tower of Pisa, tall and leaning, to carry out the demonstration, from which two lead balls were thrown, supposedly with weights x and 2x, and both would have fallen at the same rate.

At the same time, one must consider that it was comparative and observed with the naked eye. According to recent studies, it would be impossible because we cannot disregard aspects such as the variation of the different forces applied to each arm to keep balls of different weights, nor would the release time of the balls by the hands be precisely the same (Newburgh & Andes, 1995).

The history of Galileo's relationship with the Leaning Tower of Pisa and physics is an interesting subject and can be evaluated from both historical and philosophical aspects and by considering the evolution of the scientific method, going from observation to proofs via mathematical models, in addition to other aspects depending on the creativity of the teacher.

Case study: calculation of the volume and analysis of the position of the centre of mass (barycentre) of the tower of pisa

The centre of mass or the centre of gravity can be analysed based on the Leaning Tower of Pisa. Therefore, the concept of static balance and centre of gravity, mentioned above but not developed, must be understood.

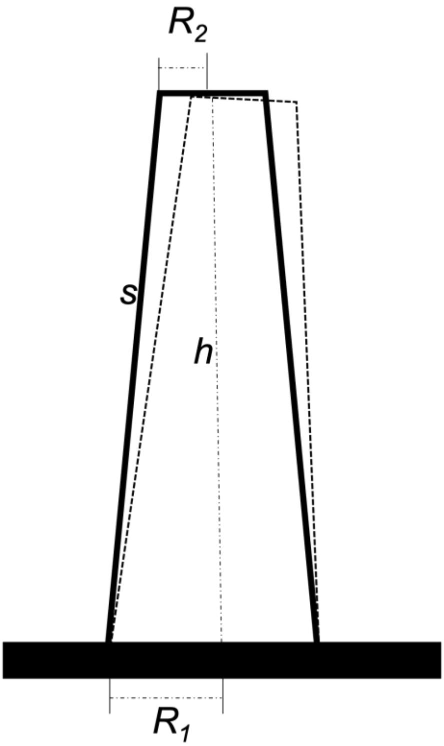
Static equilibrium refers to the equilibrium of a body that is being analysed. To reach static equilibrium, it is necessary that the sum of the forces acting on the body be zero at any point. The centre of mass is a single point in space where the sum of the weighted position vectors of all particles in the system relative to this point amounts to zero. The centre of mass and centre of gravity coincide since the acceleration of gravity is constant for the whole body extension.

During the renovations of the tower, to reduce its degree of inclination, almost 100 tons of mortar were injected into the ground. As we know that there was no change in the inclination, one of the solutions was to add extra mass to its base to reduce the angle of inclination and its centre of mass, which generated a variation in its density, making it non-homogeneous.

For an object to be in equilibrium, the projection of its centre of mass must intersect its base of support (Halliday et al., 2013). However, for teaching purposes, the tower is considered homogeneous, i.e., with a constant density (ρ). In the Tower of Pisa, at a given height, the diameter is smaller than at the base, so, as a model, it would be compared to a truncated cone (Figure 8). In this example, we used a truncated cone for the study of the maximum slope, but, in front, the tower was modelled as a cylinder (see opposite).

Figure 8

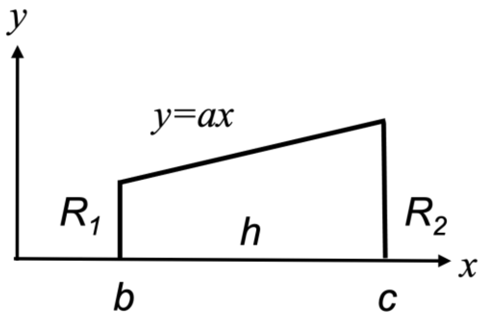
Observe the scheme of the Tower of Pisa with the dotted lines indicating the Leaning Tower and the solid line a truncated cone where the height h, the greater radius R_1, the smaller radius R_2, and the generatrix s are identified.



Note that the area A can be calculated by tilting the cone 90º, making h coincide with the x-axis, defined from b to c and with the generatrix being a function of the 1st degree of the type y=ax (Figure 9).

Figure 9

Model of the analysis of a truncated cone to calculate its volume. Consider this schematic as a slice of a three-dimensional figure.



The generator s is given by

$$(ax)^2 = s^2 = (R_1 - R_2)^2 + h^2$$

But, the revolution of the generatrix, that is, rotating the trunk area about the x-axis, we will have the volume of the cone trunk as follows

$$V = \int_b^c \pi \cdot (ax)^2 dx$$
$$V = \pi \cdot a^2 \int_b^c x^2\, dx$$
$$V = \pi \cdot a^2 \cdot \left[\frac{x^3}{3}\right]_b^c$$

$$V = \frac{\pi \cdot a^2}{3} \cdot (c^3 - b^3)$$
$$V = \frac{\pi \cdot a^2}{3} \cdot (c - b) \cdot (c^2 + cb + b^2)$$
$$V = \frac{\pi}{3} \cdot (c - b) \cdot [(ac)^2 + ac \cdot ab + (ab)^2]$$

According to Figure 9, y=f(x)=ax, indicates that f(c)=ac; therefore, ac=R_2 and, consequently, R_1=ab and h=c-b, therefore

$$V = \frac{\pi}{3} \cdot (h) \cdot [(R_2)^2 + R_2 \cdot R_1 + (R_1)^2]$$

The centre of mass is closer to the base because it contains the largest area (largest radius) and thus also the largest mass (considering the homogeneous Tower) (Hibbeler, 2016).

The calculation of the centre of mass can be performed according to the discipline involved, as it is essential for solid mechanics in civil engineering, but its position can also be interpreted in calculus and analytical geometry classes.

Calculation of the critical angle

The critical angle of the Tower of Pisa is the greatest angle of inclination that the tower can reach. The exercise to find this angle involves static balance, the centre of gravity, the similarity of triangles, and trigonometry, among other basic math concepts, and is thus highly suitable for secondary and higher education.

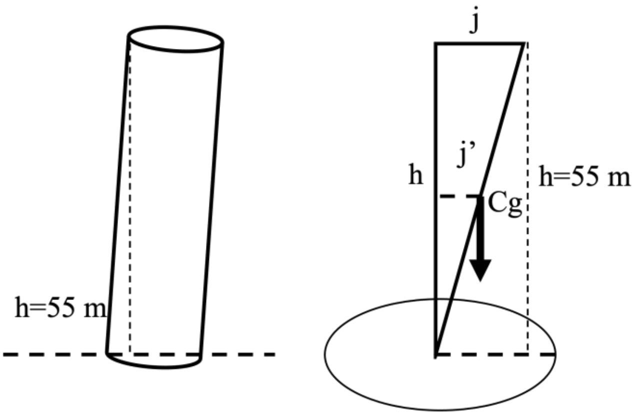
Considering the Tower of Pisa as built of a homogeneous material (the whole tower having a constant density), and, unlike the previous example model of a truncated cone, for didactic reasons here the model will be a homogeneous cylinder, with the centre of gravity at half its height. Considering a cylindrical and symmetrical tower (within the conditions imposed by the problem, the centre of gravity Cg=27.9 m.

As the vertical that intersects the centre of mass approaches the end, the greater the risk of the tower tipping over; that is, the closer it will be to its critical angle. The challenge is to lower the slope to make it safe for tourists, as it is a major tourist attraction in Italy. We know that j=4.5 m, using the similarity of triangles (Figure 10):

$$\frac{h}{j} = \frac{\frac{h}{2}}{j\prime} \Rightarrow \frac{\frac{55,86}{2}}{4,5} = \frac{27,93}{j\prime} \therefore j' = 2.41\ m.$$

Figure 10

Representation of the Leaning Tower based on the proposed problem.



The tower will reach the critical angle when the centre of mass (or centre of gravity) is at a distance *r* from line *h*. Thus, $j = j'/2$, where $j' = 3.5m$ and $j = 7m$.

To find the maximum angle, we have

$$tan\theta = \frac{opposite\ side}{adjacent\ side}$$
$$tan\theta = \frac{7}{55.86}$$

$$tan\theta = 0.1253$$

$$\theta = tan^{-1}(0.1253)$$

$$\theta = 7.42^{o}$$

The maximum angle that the tower can reach is 7.42°.

Currently, the Tower of Pisa has 3.97° of inclination with 3.17° to reach the maximum angle. Therefore, it is closer to the maximum angle than to the vertical.

Study: trigonometric ratios

An interesting activity for the study of trigonometric relationships is to use the history of the tilt of the tower throughout history to teach this subject, in this case, in high school.

To start, we use the right triangle ratios to calculate the height of the Tower of Pisa on the lower side (south side of the tower), which has been modified to try to decrease the slope. It is essential to understand the difference between the height of the tower and its length (in most cases, these two concepts are the same), but this is not the case here.

The activity consists of using the trigonometric relationships learned to find the variation in the height of the Tower of Pisa over time. As a modelling activity, the problem can be done via a survey of the tower's history (see Table 1).

To solve this problem, we need to use the trigonometric ratios, which are the relationships between the sides of a right triangle, which are:

$$sen\theta = \frac{opposite\ side}{hypotenuse};\ \ cos\theta = \frac{adjacent\ side}{hypotenuse};\ \ tg\theta = \frac{opposite\ side}{adjacent\ side}$$

Tangent and sine can be used for the process of verifying the final results.

For the year 1292, we have,

$$cos\ (1.5^{o}) = \frac{h}{55.86\ m} \Rightarrow h = cos^{-1}(1.5^{o}) = 55.84\ m.$$

For the year 1817,

$$cos\ (5.1^o) = \frac{h}{55.86\ m} \Rightarrow h = cos^{-1}(5.1^o) = 55.64\ m.$$

For the year 1990,

$$cos\ (5.5^o) = \frac{h}{55.86\ m} \Rightarrow h = cos^{-1}(5.5^o) = 55.60\ m.$$

For the year 2020:

$$cos\ (3.9^o) = \frac{h}{55.86\ m} \Rightarrow h = cos^{-1}(3.9^o) = 55.72\ m.$$

Finally, Table 2 can be provided without data for students to complete with information searches.

Table 2

Information on the degree of inclination over the years.

Year	Degree of inclination	Height (h) of the Tower
1292	1,5°	**55.84 m**
1817	5,1°	**55.64 m**
1990	5,5°	**55.60 m**
2020	3,97°	**55.72 m**

FINAL CONSIDERATIONS

After several attempts to stabilise the Leaning Tower of Pisa, only in the computer age, with computational prediction models, was it possible to initiate a stabilisation project that proved functional, at least so far.

The Tower of Pisa is one of the greatest monuments in the world, 1) since the beginning of its construction and by the medieval Roman style associated with the Baroque and 2) during construction. Interestingly, we can add the fact of its 3) inclination, a flaw that has generated considerable attention. The design time was not followed; its construction took about 199 years, involving several engineers, architects, and masons. We do not know the exact number of people involved due to a lack of complete documentation. It was a public building, and perhaps for that reason alone, it was not demolished. This may also be the reason why so much money was spent on the correction of its structures.

The construction of the Tower of Pisa shows the resilience of the human spirit, both in continuing its construction and in the attempt to correct the inclination. However, we must mention that there was no proper investigation of the project execution.

The facts cited above are splendid in terms of the history of engineering and construction, both regarding the execution and the various attempts to solve the problem of the slope, involving several areas and, perhaps most interestingly, bringing engineering closer to history. As engineering is a course with an emphasis on techniques, physics and mathematics are essential tools to improve teaching for engineers, as their teaching has been poor and impacts professionals worldwide (Alves & Aversi-Ferreira, 2019).

These problems are not prerogatives of engineering education, as the exact sciences present difficulties in quality teaching and implementing new education technologies. However, epistemology already exists for the teaching of the basic sciences, but in the case of the applied sciences, studies on how to teach these subjects is scarce.

In conclusion, the history of the Leaning Tower of Pisa has a pedagogical basis for teaching, especially in the exact sciences, as it enriches school culture and attracts an inter- and multidisciplinary character to teaching, as shown through the proposed examples. The Tower of Pisa theme can be used to implement new education technologies, such as modelling, problem-solving, and real-life problems. We not only detailed Pólya's problem-solving method to solve one problem but also provided a base for solving other problems using the same method.

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AUTHORS' CONTRIBUTIONS STATEMENTS

TAA-F conceived the presented idea, developed the theory, adapted the methodology to this context, created the models, performed the activities, and collected the data. LCSF developed the theory, adapted the methodology to this context, created the models, performed the activities, and collected the data. IC analysed the data and collected the data. RF analysed the data and collected the data. KT performed the activities and collected the data. All authors actively participated in the discussion of the results and reviewed and approved the final version of the work.

DATA AVAILABILITY STATEMENT

Data supporting the results of this research will be made available by all authors (CMP, RJCC, and IAPE) upon reasonable request.

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