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Socioepistemology of the Existence and Uniqueness Theorem in the First-order Ordinary Differential Equation

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ABSTRACT

Background: The teaching of differential equations is dominated by an excessively algebraised analytic tradition. For this reason, studies that contribute to conceptualising mathematical objects associated with the differential equation are important, particularly the existence and uniqueness theorem. **Objectives**: From its genesis, the objective is to analyse the nature of this knowledge, its epistemology from practice. We give an account of the *variational arguments*, i.e., based on practices focused on the study of change, with a predictive purpose, which allows obtaining the desired result on the differential equation: demonstrating the existence of a unique solution. **Design**: A documentary analysis is carried out from the Socioepistemological Theory of the works that marked the construction of this mathematical knowledge. **Setting and Participants**: Being a documentary-cut study, we did not have participants *stricto sensu*. **Data collection and analysis**: Our observation unit includes mathematical works as primary and secondary sources involved in constructing the theorem: its postulations, search for hypotheses and proofs. **Results**: A reconstruction of the theorem is offered, which from the arguments, characterises some practices that helped in the construction of mathematical objects. **Conclusions**: We conclude that the bounded variation, as a particular way of using change, contributed to the search or establishment of conditions for the interpretation of the solution of equations and to obtain a unique solution to the differential equation, contributions that should be key for implementations of learning situations.

Keywords: Socioepistemology; variation; differential equation; existence; uniqueness.

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Socioepistemología del teorema de existencia y unicidad en la ecuación diferencial de primer orden

RESUMEN

Antecedentes: La enseñanza de las ecuaciones diferenciales está dominada por una tradición analítica excesivamente algebrizada. Por ello, son importantes los estudios que contribuyan a la conceptualización de los objetos matemáticos asociados a la ecuación diferencial, particularmente, el teorema de existencia y unicidad. **Objetivos:** Desde su génesis, el objetivo es analizar la naturaleza de este saber, su epistemología desde las prácticas. Damos cuenta de los argumentos variacionales, es decir, basados en prácticas enfocadas al estudio del cambio, con finalidad predictiva, que permiten obtener el resultado deseado sobre la ecuación diferencial: demostrar la existencia de una solución única. **Diseño:** Se realiza un análisis documental desde la Teoría Socioepistemológica de las obras que marcaron la construcción de este saber matemático. **Entorno y Participantes:** Al ser un estudio de corte documental, no contamos con participantes en sentido estricto. **Recopilación y análisis de datos:** Nuestra unidad de observación incluye trabajos matemáticos como fuentes primarias y secundarias que intervienen en la construcción del teorema: sus postulados, búsqueda de hipótesis y demostraciones. **Resultados:** Se ofrece una reconstrucción del teorema, que a partir de los argumentos se caracterizan algunas prácticas que ayudaron en la construcción de objetos matemáticos. **Conclusiones:** Concluimos que la variación acotada, como una forma particular de utilizar el cambio, contribuyó a la búsqueda o establecimiento de condiciones para la interpretación de la solución de ecuaciones, para obtener una solución única a la ecuación diferencial, aportes que deben ser claves para implementaciones de situaciones de aprendizaje.

Palabras clave: Socioepistemología; variación; ecuación diferencial; existencia; unicidad.

Socioepistemologia do teorema da existência e da unicidade na equação diferencial ordinária de primeira ordem.

RESUMO

Contexto: O ensino de equações diferenciais é dominado por uma tradição analítica excessivamente algebrizada. Por esta razão, estudos que contribuam para a conceituação de objetos matemáticos associados à equação diferencial são importantes, em especial, o teorema da existência e da unicidade. **Objetivos**: A partir de sua gênese, o objetivo é analisar a natureza desse conhecimento, sua epistemologia a partir da prática. Damos conta dos argumentos variacionais, ou seja, baseados em práticas voltadas para o estudo da mudança, com finalidade preditiva, que permitem obter o resultado desejado na equação diferencial: demonstrar a existência de uma solução única. **Design**: É realizada uma análise documental a partir da Teoria Socioepistemológica das obras que marcaram a construção desse conhecimento matemático. **Ambiente e participantes**: Por ser um estudo de recorte documental, não tivemos participantes *stricto sensu*. **Coleta e análise de dados**: Nossa unidade de observação inclui trabalhos matemáticos como fontes primárias e secundárias envolvidas na construção do teorema: suas postulações, busca de hipóteses e provas. **Resultados**: É oferecida uma reconstrução do teorema, que a partir dos argumentos caracteriza algumas práticas que auxiliaram na construção de objetos matemáticos. **Conclusões**: Concluímos que a variação limitada, como uma forma particular de usar a mudança, contribuiu para a busca ou estabelecimento de condições para a interpretação da solução de equações, para obter uma solução única para a equação diferencial, contribuições que devem ser fundamentais para implementações de situações de aprendizagem.

Palavras-chave: Socioepistemologia; variação; equação diferencial; existência; unicidade.

INTRODUCTION

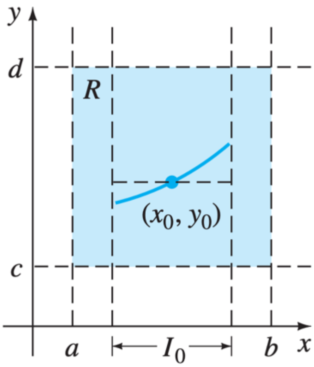
The existence and uniqueness of the solution corresponds to characteristics to be studied of some *inverse problem*, a type of problem known for determining the causes that lead to obtaining a result in a model or for determining the model based on a "cause-effect" relation (Martinez-Luaces, Fernández-Plaza & Rico, 2019; Korovkin, Chechurin & Hayakawa, 2007; and Ramm, 2005). The inverse problems are found in several everyday activities in psychology, medicine, physics, economics, mathematics, and others when inquiring about the causes or model that allow giving a known effect.

One content review of currently used college books to teaching differential equations (Fallas-Soto, 2015) addresses the existence and uniqueness theorem only as a *direct problem*; that is, it reduced the study of existence and uniqueness to proving hypotheses and a few techniques for determining the interval within which the necessary conditions were satisfied (*i.e.*, the causes and the model are known to determine the effect). An inverse problem, in contrast, entails a search for the conditions (just how necessary and sufficient?) that allow the solution to exist and be unique. Thus, we are dealing with a difference that may not be considered in the comprehension and meaning of this knowledge in the field of education.

For example, Zill and Cullem (2019) (like Braun (1993), Simmons (1993), and Zill (1997)) state the theorem of existence and uniqueness of the solution of the first-order Ordinary Differential Equation (ODE). This says: Let R be a rectangular region in the xy-plane defined by $a \leq x \leq b$, $c \leq y \leq d$ that contains the point (x_0, y_0) in its interior (Figure 1). If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution to the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

Figure 1

Rectangular region R. (Zill & Cullen, 2019)



The previous figure is only an illustration of a consolidated proof of the theorem. University mathematics – specifically, the teaching of differential equations– is dominated by an overly-algebraised analytical tradition (Barros & Kato, 2016; Dana-Picard & Kidron, 2008; Moreno, 2006). On the other hand, if, as teachers, we could consider a conceptualisation of the geometric focus in the learning of differential equations and their solution, confront the difficulties inherent in making long-term predictions, and evidence the role played by initial values, among other elements (Karimi Fardinpour & Gooya, 2017) we could favour the learning of this knowledge in our students. The literature has reported that just like considering procedural mathematics to solve differential equations, the conceptualisation of the mathematical objects associated with the equation as a model should also take on importance in the lesson (Artigue, 1992; Buendía & Cordero, 2013; Camacho-Machín & Guerrero-Ortiz, 2015; Fallas-Soto & Cantoral, 2016; Morales & Cordero, 2016; Rasmussen, Zandiech, King & Tepo, 2009; Stephan & Rasmussen, 2002).

Differential equations emerged and developed mainly as predictive models of the behaviour of certain phenomena of physical movement, with the goal of mathematizing nature in Galileo's sense. *Variation* and *change* play important roles in this phenomena study as they do in its mathematization: variation encompasses a quantification of change in the variables of a phenomenon [but] not just any variable [only] those that are causally related (Cantoral, Moreno & Caballero, 2018).

From the genesis of this inverse problem, the objective is to analyse the nature of this knowledge, its epistemology (based on practices). Accepting that school mathematical knowledge is relative, we assure one that the results obtained will contribute to understanding this subject for its teaching and learning. We give an account of the *variational arguments*, that is, based on practices focused on the study of change, with a predictive purpose, which allows obtaining the desired result on the differential equation: demonstrating the existence of a unique solution.

This led us to perform a documental analysis of the ideas in the original mathematical works that spawned the theorem. We consider a set of reflections on the essence of this knowledge based on the evolution of its results and the collection of anecdotes over time that explains the nature of its practices, its social construction (opportunities and restrictions that emerge when answering when, where, who and why appear in the construction of that knowledge), and its institutional diffusion as specialised mathematical knowledge.

Another contribution of this work, and that we will defend during the development of the reading, is that to study the existence and uniqueness of the solution of differential equations, a new notion is merited that makes specific use of the variation establishing conditions as boundaries, on the change in the functional variables with respect to time. We call this *bounded variation*.

THEORETICAL BACKGROUND

While the term "constructivism" has a broad range of meanings, it is deemed to affect students' learning in educational circles positively. At least two principal types of constructivism are recognised: cognitive (Piaget, 1976) and social (Vygotsky, 1986). While both are constructivist in nature, the main concept we are interested in analysing considers a combination of the two: ideas are constructed cooperatively through an individual's experiences in and on the environment. In this sense, our work is based on principles of cognition (*i.e.*, the interiorised actions of the individual exposed to a situation) and the social (*i.e.*, internalised activities like the rational coordination of an individual's actions in response to situations). The Socioepistemological Theory of

Mathematics Education (STME) holds that these elements are organised through socially shared practices in the understanding of shared uses and meanings (that enable the group's goal when confronting a task, as the most basic link in this theoretical explanation, since it is normed and maintained socially, and comprises the rationality of peers) (Cantoral, 2019), that is, an epistemology of practices.

The organisation of practices happens because of a norm, precisely when a group creates a language and communicates by itself. According to Cantoral (2019), this is a socially shared practice where the relationships between discourse and temporally located knowledge are investigated, thus, trying to find how, in each practice, the subject and the object of knowledge are constituted (vision of the norm according to Foucault, taken from (Escolar, 2004)).

Therefore, we do not analyse the proof of the theorem in a paper but incorporate, as well, the relations among people in each epoch and under certain circumstances. As Cantoral (2019) pointed out regarding this type of research problem, a double decentration of the object and its didactic models is required to focus attention on the *practices* that aided in their construction. Metaphorically speaking, the aim is to focus on the process of mathematization rather than the mathematical result or product.

We begin by considering the four theoretical principles on which the STME acts, always in an articulated manner, for the social construction of knowledge (Cantoral, 2019; Cantoral, Montiel & Reyes-Gasperini, 2015):

- *Contextualised rationality* alludes to the notion that the relationship with knowledge is a contextual function. This principle allows us to recognise, privilege, and potentiate diverse types of rationality related to the reality in which an entity finds her/him/itself or themselves at a given moment and in each place from which knowledge will be constructed.
- *Epistemological relativism* recognises that the validity of knowledge is relative to the entity in its cultural group. School mathematics can be perceived in many ways as being worked, constructed, or developed, conceiving that the validity of knowledge is relative to the entity in the cultural group from which it emerges based on the contextualised rationality that she/he/it possesses.

- *Progressive resignification* holds that meaning is not static but relative, functional, and contextual. Interaction with diverse contexts and the evolution of the life of an individual or group will re-signify the knowledge constructed up to that moment and enrich it with new meanings.
- *Normativity of social practice* is the principle that makes it possible to achieve the meaning of mathematics through use, proposed in an organisation of socially-shared practices (the triad of actions, activities, practices) that are regulated and normed, respectively, by practices of reference and social practice.

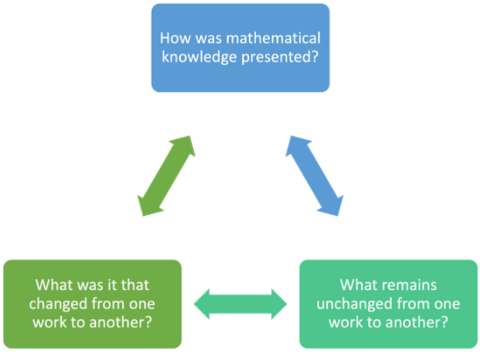
Another theoretical element to consider is the *problematizing mathematical knowledge,* characterised by *historicization*; that is, connecting with an epistemology situated in the time and study of a social history that constitutes it, and *dialectization* it in the confrontation that is necessary for understanding alternative meanings and rationalities in what is assumed as a constructed knowledge. The historicization "includes the research on the sociocultural context of the author at the time the knowledge was developed, the author's concerns about the problem that was being solved, the mathematical work, and a reconstruction of that work" (Hinojos-Ramos, Farfán & Orozco del Castillo, 2020, p.1163). The dialectization discusses and dialogues to identify a relativism and rationality of knowledge, significances, by evidencing differences between scholarly mathematical knowledge and different scenarios, whether technical, popular, or scientific, accepting contradictions and confrontations of ideas. In fact, this work corresponds to a problematization of mathematical knowledge intending to continue growing.

METHODOLOGY

A documentary analysis of the mathematical works that were building this theorem is carried out, demonstrating conjectures. Each text analysed is a product with a history (so-called social history) that emerges upon critically exploring the context in which it was created, viewing it as an object of diffusion considering the public to which it was directed, pondering a certain didactic intentionality, and conceiving it as part of a more global intellectual expression with respect to how it influenced other contexts (Espinoza-Ramírez, Vergara-Gómez, Valenzuela-Zuñiga, 2018). We also carried out a comparison of the works closest to the theorem to answer the following questions (Figure 2).

Figure 2

Comparison of the mathematical works



For the question "how was knowledge represented?" we described the sections that precede and follow the theorem in the relevant mathematical works. For the questions posed, it was important to locate and analyse the elements that were maintained and those that changed in each text –or re-editions– to identify similarities and differences in their respective conceptual evolutions.

Therefore, a reconstruction of the existence and uniqueness theorem supported by the visualisation of arguments is provided. The proofs are presented as direct problems since they are shown as finished products. Still, the rationality and how the theorem evolved reveal characteristics as inverse problems by looking for the minimum number of *sufficient, but not necessary, conditions* that ensure the existence and uniqueness of solving differential equations.

In the search for sources, Picard (1886) mentions an important antecedent that Cauchy and Moigno (1844) established with respect to the theorem, one later simplified by Lipschitz (1880) (Figure 3).

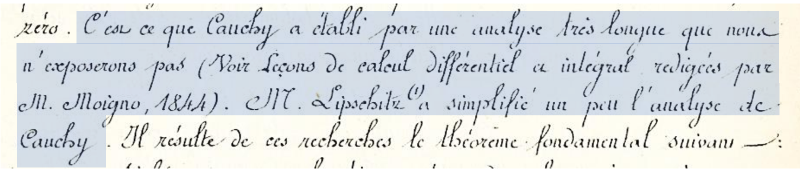
We also examined Peano's work since his name appears in university textbooks about the theorem of existence. The texts that formed our working material, or unit of study, include the following primary sources and textbooks:

- *Cauchy and Moigno (1844), Leçons de Calcul Différentiel et de Calcul Integral*

- Lipschitz (1880), *Lehrbuch der Analysis: Differential und integralrechnun*
- Lipschitz (1868), *Disamina della possibilità d'integrare completamente un dato sistema di equazioni differenziali ordinarie*
- Peano (1886), *Sull' integrabilità delle equazioni differenziali di primo ordine*

Figure 3

Picard's recommendation. (Picard, 1886, p. 293)[1]


zéro. C'est ce que Cauchy a établi par une analyse très longue que nous n'exposerons pas (Voir Leçons de calcul différentiel et intégral rédigées par M. Moigno, 1844). M. Lipschitz(1) a simplifié un peu l'analyse de Cauchy. Il résulte de ces recherches le théorème fondamental suivant :

The principles of Socioepistemological Theory are manifested in each stage of the problematizing of mathematical knowledge. Achieving an adequate orientation when analysing texts from earlier periods requires contemplating the context provided by details of the epoch in which they appeared and were developed to understand the rationality of the authors who created and utilised those writings (contextualised rationality). This further demands recognising and accepting the relativity of knowledge – contrary to the absolutism of unique truths – as an interpretation of mathematics with respect to what we know today (epistemological relativism). During our examination of these works, supported by visualisation, we developed a dialectic for confronting evolution (before-after) and reflected on the arguments that explain both the use and meaning of knowledge (progressive resignification) that, in the end, help by inferring practices in the construction of knowledge (normativity of practice).

[1] Own translation: It is that Cauchy establishes a very broad analysis that we will not present here (see lessons on differential and integral calculus by M. Moigno, 1844). Mr. Lipschitz simplifies Cauchy's analysis somewhat.

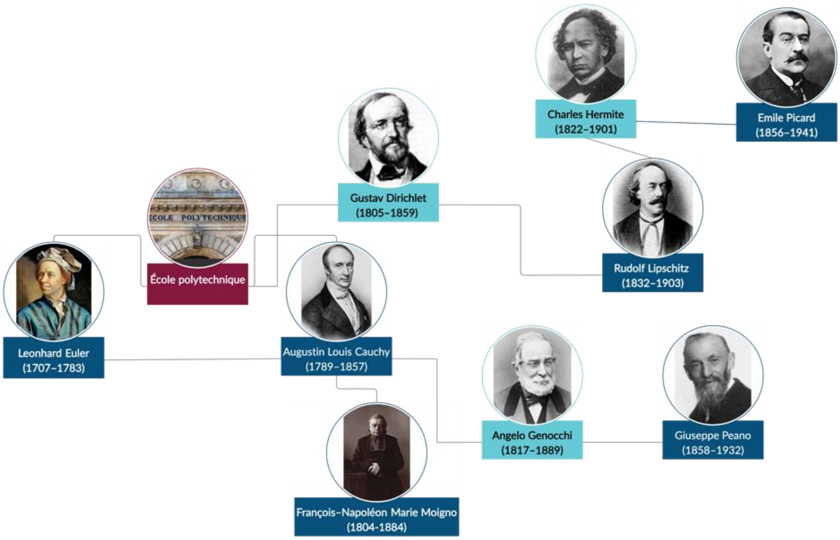
RESULTS AND ANALYSES

Rationality and context of the texts

Behind each work we analysed, there lies a network of contributions and influences of other people and their texts that impacted the legacy of knowledge; each text is linked to a very wide network. Figure 4 highlights the authors who directly influenced the texts we examined and played key roles in developing the result (*i.e.*, the postulation and demonstration of the theorem) to identify this conceptual weaving.

Figure 4

Scheme representing relations among mathematicians



As is well known, Cauchy did not only write books on mathematics; but also on the French Revolution, socio-political movements in the France of the 1830s, and, to some degree, on his Jesuit principles (which led him to the priest, journalist, and mathematician, Moigno, with whom he wrote some books). This last forced him to abandon the country and go into exile after refusing to swear loyalty to Louis Philippe I. However, this did not impede him from continuing to develop his works. His exile led him to Italy, specifically the University of Turin, where the mathematician Genocchi (a professor of

Peano, who was not only his student but also an assistant and disciple) incorporated ideas from his works into his courses on mathematical analysis.

The construction of the mathematical analysis proposed by Cauchy responded to the mathematical concerns of the German Academy. Certain features of his work reveal his declared intention to abstain from any historical reference (showing interest only in Euler's work) and avoid geometric examples and the support of analogies of a phenomenological nature in his procedures (Dhombres, 1985). Cauchy's approach reformulated the rationality of mathematical works of the so-called Classical Analysis type, clearly based on physical phenomena of nature and geometric arguments. This contrasts with the current practice where algebraic analysis predominates. In books of Classical Analysis, the mathematization of nature ceded its place to an algebraization of functions and of the infinitesimals that emerged as useful for formalisation.

According to the report in Youschkevitch (1981), it was no easy matter to present new ideas, forms of reasoning, or ways to propose problems of existence to France's Council of Public Instruction. One innovative contribution of Cauchy's course is that it was the first to include a theorem of the existence of the solution of general, first-order differential equations, developed in lessons 26 and 27 in (Cauchy & Moigno, 1844).

Meanwhile, Dirichlet (Lipschitz' professor), while working on the series' convergence under Fourier's supervision, presented subtle challenges to Cauchy's proof (Lakatos, 1980). His work perhaps reflects problems inherited by German mathematicians from the French School of Mathematics. Dirichlet also served as an adviser on the doctoral dissertation presented by Lipschitz, who later resumed on Cauchy's theorem of existence and uniqueness.

Finally, Hermite, Picard's father-in-law who lived on the French/German border, was a close friend of Lipschitz, and the two maintained an intense correspondence consisting of 148 letters and 9 postcards between August 19, 1877, and July 14 1900, a few months before Hermite's death (Goldstein, 2018). In those missives, Lipschitz shared his work entitled *Lehrbuch der Analysis,* to which Picard later referred. This brief history allows us to infer the existence of a network of collaborations, debates and, finally, precisions, regarding the theorem that interests us. The following section presents a rational reconstruction of this network of meanings and procedures.

Cauchy and Moigno's work

First, Cauchy and Moigno (1844) considered the equation $y' = f(x, y)$ with an initial value (x_0, y_0). They hypothesised that the functions $f(x, y)$ and $\frac{\partial f}{\partial y}$ will be continuous in the vicinity of the initial value. This aspect requires an analysis of its rationality to clarify their affirmation. The proof of the existence of the solution begins with the method that Euler utilised to determine the solution of differential equations through polygonal approximations to a solution (Figure 5). In that case, the two practices that predominated were approximation –to determine numerical values that approach the solution of the equation– and comparison between one state and a later one, as can be appreciated below. Attention focused on the form of articulating the *approximation* with the *comparison* to visualise the meaning in the proof of the theorem. However, the problem of continuity in the hypothesis demands an even more detailed technical analysis.

Figure 5

Euler's method. (Cauchy, Moigno, 1844, p. 386)

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n valeurs correspondantes de $y, y_1, \ldots, y_{n-1}, Y$, à l'aide des équations

$$\begin{aligned} y_1 - y_0 &= (x_1 - x_0) f(x_0, y_0), \\ y_2 - y_1 &= (x_2 - x_1) f(x_1, y_1), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ Y - y_{n-1} &= (X - x_{n-1}) f(x_{n-1}, y_{n-1}), \end{aligned}$$

en éliminant $y_1, y_2, \ldots, y_{n-1}$, on obtiendra une valeur de Y de la forme

$$Y = F(x_0, x_1, x_2, \ldots, x_{n-1}, X, y_0),$$

The differential equation $y' = f(x, y)$ represents the slope of the tangent at any given point of the solution, setting out from an initial value (x_0, y_0). Using this value and the slope $f(x_0, y_0)$ allow us to determine the equation of the tangent to the solution that passes through that point (Figure 6).

Figure 6

First iteration of Euler's method

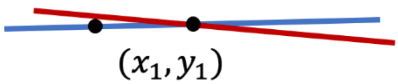
$$y - y_0 = (x - x_0)f(x_0, y_0)$$

This tangent line represents, in Lagrange's sense, the linear function that best approximates the curve at the solution point, considering its initial value. Next, a second point (x_1, y_1) on the line is chosen, one that is also very close to (x_0, y_0). Once again, a differential equation is used with this point to determine the line that passes through (x_1, y_1) with slope $f(x_1, y_1)$ (Figure 7).

Figure 7

Second iteration of Euler's method

$$y - y_1 = (x - x_1)f(x_1, y_1)$$



Continuing with this method, as many points as are required are obtained. What is obtained, in fact, is a discrete approximation of the solution of differential equations if the differences are finite (as in Figure 8) but continuous if the differences are infinitely small. The prediction comes into play in this strategy upon asking what will occur *in the near future*. However, at this moment, no reflection on the uniqueness of the solution exists, at least not explicitly. If the values converge, then the limit is a function that is the solution of the equation. Consequently, if the values do not converge, this does not imply anything about the existence of the solution, so there might be a solution for the equation and no convergence with the method.

Figure 8

nth iteration of Euler's method

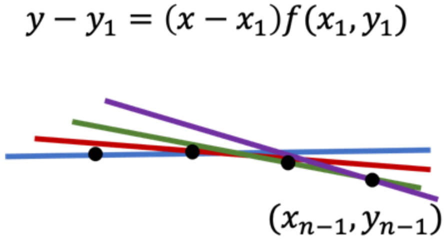
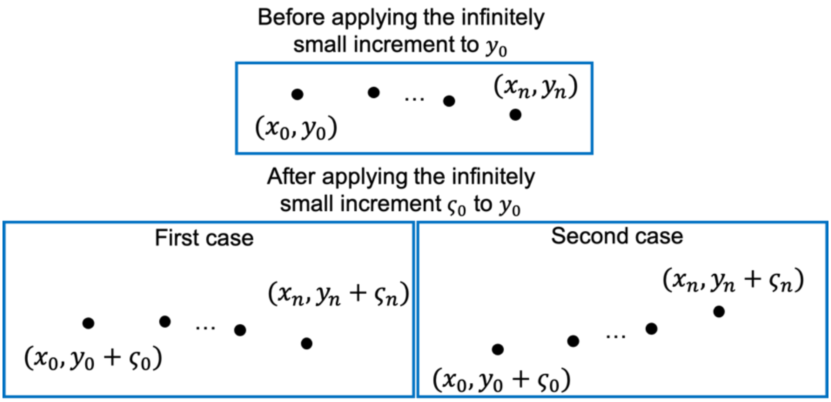


Figure 9

By applying an infinitely small increment to the initial value



The convergence of the values obtained will depend on guaranteeing the existence of the solution. Cauchy and Moigno (1844) utilised the infinitely small (infinitesimal) increase as follows: if an infinitely small increase is made ς_0 to y_0, then y_n will have an increase of ς_n. For this to converge to the desired solution, the latter increase must be equally small as ς_0, and fineness begins on *bounded variation*. In contrast, if the infinitely small change, ς_0, is applied to y_0 and the change, ς_n, in y_n is not equally small as ς_0, then Cauchy and Moigno (1844) affirmed that it would not converge with the solution offered by the method (see Figure 9). This evidence the role of *small variation* in the analysis of the convergence of the approximation as, for example, the use of the infinitely small increase in its different orders.

Suppose that, with the change ς_0 to y_0, the value of y_1 is affected by an increase, ς_1. Two equalities are obtained, one without the increase, the other with ς_0:

$$y_1 - y_0 = (x_1 - x_0)f(x_0, y_0)$$

$$y_1 + \varsigma_1 - (y_0 + \varsigma_0) = (x_1 - x_0)f(x_0, y_0 + \varsigma_0)$$

The second equality is subtracted from the first to produce:

$$\varsigma_1 - \varsigma_0 = (x_1 - x_0)[f(x_0, y_0 + \varsigma_0) - f(x_0, y_0)]$$

Utilising the hypothesis that $\frac{df}{dy}$ is continuous in a closed interval, ensures that the difference is bounded by a constant, M, such that:

$$[f(x_0, y_0 + \varsigma_0) - f(x_0, y_0)] < M\varsigma_0$$

This is where one specific study of variation is evidenced, as *bounded variation* appears as a condition on the slopes of tangents to guarantee, in this case, the convergence and existence of the solution. This is fundamental for constructing Lipschitz's condition – discussed below –to ensure the solution's uniqueness. As a result of this, we have

$$\varsigma_1 - \varsigma_0 < (x_1 - x_0)M\varsigma_0$$

Hence

$$\varsigma_1 < \varsigma_0(1 + (x_1 - x_0)M) < \varsigma_0 e^{(x_1 - x_0)M}$$

Where ς_1 will be as small as possible, depending on the value of the difference $(x_1 - x_0)$. This leads to testing for y_n and studying the differences of the $y_m - y_{m-1}$ type for m natural, between 1 and n, where an increase of $\varsigma_m e^{(x - x_m)M}$ is attributed to y_m. This proves the convergence of a function $y = F(x)$, which could be a solution to the equation.

Turning to the test of continuity, the following equalities are taken, where A is an average of $f(x_n, y_n)$, while Θ, θ, and θ_1 are values between 0 and 1.

$$y_1 - y_0 = (x_1 - x_0)f\big(x_0 + \theta_1(x_1 - x_0), y_0 \pm \Theta A(x_1 - x_0)\big)$$

Due to the continuity of the function, f, considered to ε infinitesimal, we have

$$f\left(x_0 + \theta_1(x_1 - x_0), y_0 \pm \Theta A(x_1 - x_0)\right) < f(x_0, y_0) + \varepsilon$$

Therefore

$$y_1 - y_0 < (x_1 - x_0)f(x_0, y_0) + (x_1 - x_0)\varepsilon$$

Again, the result depends on the value of the difference $x_1 - x_0$. Finally, to prove that the differential equation is satisfied, we use the equality

$$F(x) = y_0 + (x - x_0)f\left(x_0 + \theta(x - x_0), y_0 \pm \Theta A(x - x_0)\right)$$

Where,

$$F(x + h) = y_0 + (x + h - x_0)f\left(x_0 + \theta(x + h - x_0), y_0 \pm \Theta A(x + h - x_0)\right)$$

Concluding that

$$F(x + h) - F(x) = hf(x_0 + \theta h, y_0 \pm \Theta Ah)$$

Which is equivalent to

$$F'(x) = f(x, F(x))$$

Thus, the differential equation is satisfied. In conclusion, Euler's method guarantees the existence of a continuous solution by means of approximation.

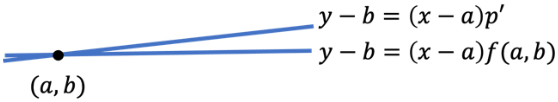
Peano's work

Peano (1886) utilised small variations, but in another direction, by employing a double inequality. The existence of a solution can be proven using differential inequalities ($y' < f(x,y)$ and $y' > f(x,y)$, with the help of the so-called super- and sub-solutions. Peano's main proposal was to determine a solution Y_1 that bounds superiorly to the solution Y of the differential equation; with a solution, Y_2, that bounds inferiorly to the solution Y. However, this argument is insufficient to guarantee the uniqueness of the solution, though it does bound it.

For the differential equation $y' = f(x,y)$, he considers the hypothesis that the function $f(x,y)$ is continuous with the initial value (a,b). Like Cauchy and Moigno's (1844) procedure, the Euler method was used, but in this case, it was based on constructing lines with slopes greater (super-solutions) or lesser (sub-solutions) than the tangent lines. For the case of the super-solutions, this assumes the existence of a p' greater than $f(a,b)$; thus, obtaining a line with a slope greater than that of the tangent line (see Figure 10).

Figure 10

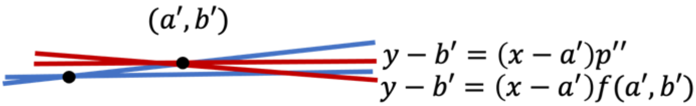
Line with a slope greater than the tangent line at the first point



After that, a point (a', b') on the line with slope p' near the initial value is taken. The line with slope $f(a', b')$ is determined with this new point, and another value, $p'' > f(a', b')$, is considered to bind the approximation superiorly (see Figure 11).

Figure 11

Line with a slope greater than the tangent line at the second point



Next, using the line with slope p'', the point (a'', b'') is obtained near the point (a', b'). This method is repeated to ensure that the infimum of the super-solutions corresponds to a function Y_1. A similar procedure is applied to determine the sub-solutions, assuming lower values for the slope than $f(a, b)$. This permit the construction of the function Y_2 as the supremum of the sub-solutions. Peano later demonstrated that both Y_1 and Y_2 are solutions of the given differential equation. Suppose then that $Y_1 = F(x)$ and that $f(x_0, F(x_0)) = m$ for x_0, a specific value of x. This shows that $F'(x)$ is equal to m to satisfy the differential equation. The function $\varphi(x) = F(x_0) + (x - x_0)(m - \varepsilon)$ is constructed with a positive ε that is sufficiently small for the line $y = b + (x - a)p'$ with $p' > m$ to be greater than $\varphi(x)$ and, since Y_1 is the inferior limit of these lines, we have

$$Y_1 = F(x) > \varphi(x) = F(x_0) + (x - x_0)(m - \varepsilon)$$

Therefore

$$\frac{F(x) - F(x_0)}{x - x_0} > m - \varepsilon$$

(1)

Finally, the function $\psi(x) = F(x_0) + \delta + (x - x_0)(m + \varepsilon)$ is constructed with δ and an ε that is positive and infinitely small. Upon performing the subtraction $\frac{d\psi}{dx} - f(x, \psi)$ we obtain

$$m + \varepsilon - f(x, F(x_0) + \delta + (x - x_0)(m + \varepsilon))$$

Which represents a positive quantity. Therefore, $\frac{d\psi}{dx} > f(x, \psi)$ and, since $F(x)$ is the lower limit that satisfies that condition, $F(x) < F(x_0) + \delta + (x - x_0)(m + \varepsilon)$; hence

$$F(x) \leq F(x_0) + (x - x_0)(m + \varepsilon)$$

That is

$$\frac{F(x) - F(x_0)}{x - x_0} < m + \varepsilon$$

(2)

Due to conditions (1) and (2), we see that $F'(x) = m$ satisfies the differential equation. The same procedure is followed analogously for Y_2.

Lipschitz's work

The Lipschitz's contributions (1868, 1880), the proposal is that a function $f(x, y)$ satisfies the condition

$$|f(h, k) - f(h, l)| < M \cdot |k - l|$$

Where M is currently known as the Lipschitz constant, and the *bounded variation* corresponds to the limits placed on the change and what it implies in the model, concluding that this condition is sufficient, but not necessary, for guaranteeing the existence and uniqueness of the solution. Lipschitz contemplates this condition in his works (1868, 1880) and deepens Cauchy's method for systems of differential equations. As shown above, Euler's method is utilised assuming that the differential equation $y' = f(x, y)$ with initial value (x_0, y_0) has two solutions; hence, there would be two possible, but clearly different, paths of the tangent lines being determined, though

Lipschitz's condition restricts this so that only one path can exist for these tangents. The solution, then, has uniqueness.

Figure 12

Lipschitz's condition. (Lipschitz, 1880, p.501)

Stetigkeit einer Function einer Variable gemacht wurde; der Kürze halber werde der absolute Werth einer Grösse w durch das Zeichen $[w]$ ausgedrückt. Es sollen der absolute Werth von (2) und von (3) die Eigenschaft haben, falls für eine hinreichend kleine Grösse δ die Ungleichheiten

(4) $$[\Delta x] < \delta, [\Delta y] < \delta, [\Delta z] < \delta$$

erfüllt sind, immer unter dieselbe beliebig kleine Grösse λ herabzusinken, oder die Ungleichheiten

(5) $$[f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)] < \lambda,$$

(6) $$[g(x + \Delta x, y + \Delta y, z + \Delta z) - g(x, y, z)] < \lambda$$

zu befriedigen. Eine andere Voraussetzung, welche ebenfalls bei den erfahrungsmässig vorkommenden Functionen in der Regel erfüllt ist, bezieht sich auf die Differenzen (2) und (3) bei je zwei der Mannigfaltigkeit K angehörenden Werthsystemen, in denen der Werth der Variable x derselbe oder Δx gleich Null ist; sie besteht darin, dass es endliche positive Constanten $c_{11}, c_{12}, c_{21}, c_{22}$ giebt, für welche die Ungleichheiten

(7) $$[f(x, y + \Delta y, z + \Delta z) - f(x, y, z)] < c_{11}[\Delta y] + c_{12}[\Delta z],$$

(8) $$[g(x, y + \Delta y, z + \Delta z) - g(x, y, z)] < c_{21}[\Delta y] + c_{22}[\Delta z]$$

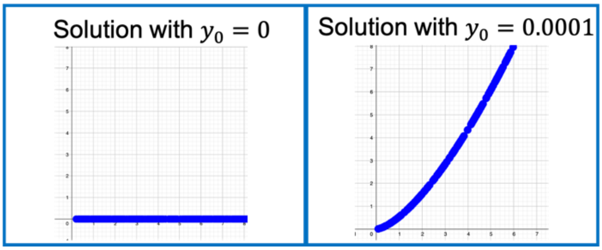
One interpretation of this problem is that if an infinitely small change is applied to the value y_0, then the difference between the slopes (before and after applying the change) will be bounded. An example of non-uniqueness is often used to visualise the fact that Lipschitz's condition is not satisfied. As an additional datum, consider the following example, given by the differential equation $y' = y^{\frac{1}{3}}$ with initial value $x_0 = 0$ and $y_0 = 0$. At least two solutions are obtained, whose analytical expression is as follows:

$$y = 0, \qquad y = \frac{2}{3}\sqrt{\frac{2}{3}x^3}$$

Thus, the solution is not unique for this initial value. If we apply Euler's method in a specific case, supposing a difference of 0.001 between x_m and x_{m+1} (with m between 0 and n as the number of iterations) and considering the initial value (0,0), the solution obtained is $y = 0$. In contrast, if we consider the initial value $(0,0+\varsigma)$ with ς sufficiently small (for this case, we consider a $\varsigma = 0.0001$), then the solution tends towards $y = \frac{2}{3}\sqrt{\frac{2}{3}x^3}$ (Figure 13). For example, this differential equation satisfies Peano's hypothesis; however, uniqueness is not satisfied.

Figure 13

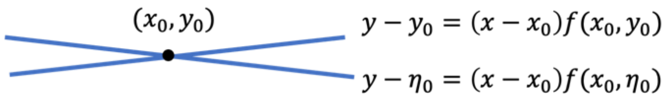
Numerical approximations of the solution of differential equations around (0,0), given by Euler's method



What happens here is that, from the first iteration, considerable changes are obtained in the slope of the tangent in each approximate value. Therefore, the property of bounded variation is not satisfied; that is, for two infinitely close initial conditions, the behaviour of each solution should be virtually the same, with no singularities, as it is part of the sensitivity of the initial value.

Figure 14

The existence of the two solutions for one equation



Consider, for example, the point (x_0, y_0) and another infinitely close point (x_0, η_0), such that $\eta_0 = y_0 + \varsigma_0$, with ς_0 being infinitely small. Using these initial values, lines can be determined for each point (as in Figure 14):

$$y - y_0 = (x - x_0) f(x_0, y_0),$$

$$y - \eta_0 = (x - x_0) f(x_0, \eta_0)$$

As shown in the representation in Figure 14.

Upon studying the difference between these two sufficiently close initial line values, we see that their slopes differ, as $|f(x_0, \eta_0) - f(x_0, y_0)|$. As a result, by satisfying Lipschitz's condition, these differences for each determined point are bounded by

$$|f(x_n, \eta_n) - f(x_n, y_n)| < M|\eta_n - y_n|$$

Where the value of M can be obtained by determining a limit of $\frac{\partial f}{\partial y}$. If the value of the change $\varsigma_n = \eta_n - y_n$ is infinitely small as ς_0, then we have $|f(x_n, \eta_n) - f(x_n, y_n)|$ is infinitely close to zero, a sufficient condition for obtaining a unique inclination of the lines for values that are very close. If Lipschitz's condition is not satisfied, uniqueness cannot be guaranteed. With $f(x, y)$, a continuous Lipschitz function, existence and uniqueness are ensured about the initial value. This is another example of the role that the study of bounded variation plays and could provide the basis for didactic designs suitable for university students.

The dialectic between university knowledge and knowledge in use in the mathematical texts analysed

Studies of this kind make it possible to develop the dialectic between institutionalised knowledge and knowledge situated at a certain moment of its historical-conceptual evolution, which in the present case corresponded to its genesis. In institutionalised university knowledge, the theorem is presented, as we have shown, as a direct problem that consists in proving a hypothesis. For the meaning of the theorem as *knowledge in use*, in contrast, we set out from its genesis, presenting it as a product of studying an inverse problem, namely, determining the conditions for ensuring the existence and uniqueness of the solution. The first approach reflects efforts to "massify" higher education, where knowledge is often presented as if finished due to the enormous amounts of content to be studied. The following table displays a synthesis of the use of the mathematical objects involved in demonstrating the theorem for the differential equation $y' = f(x, y)$ with initial value (x_0, y_0). One column shows university knowledge, the other its genesis.

Table 1

The dialectic between university knowledge and knowledge in use in the mathematical texts analysed

Object	University knowledge	Knowledge from its genesis
$\boldsymbol{f(x, y)}$	Analytic expression that corresponds to the derivative (operationally) of the solution.	Signification of the derivative of the solution. Change, the slope of the tangent at each point of the solution.
Initial value $(\boldsymbol{x_0}, \boldsymbol{y_0})$	Determine a specific solution or interval that satisfies the hypothesis with the initial value.	Application of infinitely small changes to the initial value to compare the approximations to the solution of initial values that are "very" close.
Lipschitz' condition, or continuity of the functions $\boldsymbol{f(x, y)}$ **and** $\boldsymbol{\frac{\partial f}{\partial y}}$	Corroborate satisfaction of the conditions with the differential equations given and the initial value established.	Search for conditions that ensure a behaviour and the existence and uniqueness of the solution. Counterexamples to prove the hypothesis (abduction).
Numeric method	Picard's iterative successions method is used, which, in its convergence, helps determine an analytic expression of the solution.	Used Euler's method to approximate the unknown (solution) to what is known (line constructed based on a point and a slope).

Based on its genesis and the role of visualisation in reconstructing the theorem, this knowledge makes it possible to identify practices associated with the search for a hypothesis that ensures the existence and uniqueness of the solution of differential equations and then demonstrates it. The practices shown in the following table were not obvious because they were blocked or concealed by the discourse of school mathematics.

Table 2

The role of practices in constructing and demonstrating the theorem of existence and uniqueness

Practice	Characterisation	Role in demonstrating the theorem
Approximate	Obtain a result as close as desired to the exact value, conserving control over closeness.	This practice appears in using the numerical method to approximate the solution of linear differential equations when an initial value is known.
Compare	Associated with the action of establishing differences between proximal states. At least two states are used when comparing.	In this case, it appears as the way to analyse the solution before and after applying an infinitely small change from initial values.
Conjecture	Making judgments about something based on the information obtained, in this case, for predictive purposes by building conditions to obtain a result	This practice appears in the study of limits on the change, intending to establish a hypothesis that ensures the existence and uniqueness of the solution of the differential equation.
Predict	Construction of affirmations with rationality that indicate that certain events are going to occur.	Construction and proof of the theorem to use it as a predictive model, continuing with the discovery of the subject.

The present study characterises specific forms of using change, where it is necessary to limit a difference of the first order of variation; that is, to limit $f(x_0, y_0 + \varsigma_0) - f(x_0, y_0)$, when (x_0, y_0) is the initial value given, and ς_0 is the infinitely small change. Variation corresponds to a study, interpretation, and quantification of the change that runs from building a differential equation to determining its solution. In the meaning, comprehension, and use of the solution, especially, small variation is evidenced as the study of what a small change (from a relativist view) can cause, and bounded variation as the study of establishing limits on change and obtaining a desired result or explain the result obtained.

Small variation offers a dynamism of the initial value that, occasionally, is used only to obtain a specific solution. Comparing specific solutions provides a global understanding of the family of solutions of differential equations. This study of small variation is contemplated within bounded variation for the construction and meaning of Lipschitz's condition, as was evidenced in the reconstruction of the theorem. Above all, "small variation" is relative according to the context used.

CONCLUSIONS

This study, from an STME perspective, broadens our knowledge of the theorem of the existence and uniqueness of differential equations and gives a plausible meaning that requires only ideas of calculus but, above all, shows the type of *practices* that come into play to justify it; on the one hand, existence, on the other, perhaps more complex, uniqueness. These findings are the fruits of an adequate problematization of mathematical knowledge.

By studying the conceptual framework of the theorem, we reflect on the complexity of conducting research in the history of mathematics. Many times, the facts are presented linearly according to the chronology. Still, particularly in this study under the socio-epistemological approach, the careful study that social interactions entail until turning them into cultural heritage is evident.

In the field of Mathematics Education, conceptual history is studied through these works based on a social construction (study of situated and shared practices). This contributes to reflections on the teaching and didactics of these topics. The aim is not to reproduce history but to extract those elements that, specifically in this paper, are reported through results to reflect on a change of relations with knowledge. The role of the mathematical object as knowledge from its genesis (Table 1), and the role of practice in demonstrating the theorem (Table 2), are characteristics used for other activities of human knowledge (popular, technical, scientific) that could be developed at various educational levels. Not only in the field of differential equations but also, for example, in the study of the back-and-forth trajectory of drones, the emptying of recipients, or medical situations, among many others.

These results reveal diverse rationalities that function in designing learning situations for differential equations or, more generally, for studying change in inverse problems. Reconstructing these meanings contributes to understanding the theorem of existence and uniqueness from a variational perspective –visual, numerical, and analytic– in addition to the algebraic approach that predominates in university textbooks and the mathematical works analysed.

One fundamental result consists in constructing this theorem and demonstrating it as a bridge between two practices of reference: the mathematization of nature and formalised mathematical theorisation, where it is institutionalised as one of the most important theorems of differential equations. In this regard, two events that brought about and explained the genesis of this problem were: (i) searching for a formalisation of the demonstration of the theorem and (ii) determining the minimum number of hypotheses that guarantee existence or even existence and uniqueness.

One well-known issue in the teaching of differential equations involves thinking of them as a set of algebraic methods that help integrate and determine a solution; that is, integration is used as an inverse method to derivation to obtain solutions. Aided by historization, we showed that demonstrating this theorem can be approached through variations and is easier to understand or more intuitive. We further showed that it is possible to obtain variational arguments to determine its solution from the equation, together with the visual interpretation of the differential equation as an inverse problem of the tangent to the given curve, which generated more visual representations and arguments of the variational type.

The use of visualisation with the structure of the inverse problem revealed the importance of the initial value in differential equations (existence) by establishing conditions for the change to obtain the desired solution (convergence of the solution and uniqueness). That visual reconstruction showed Euler's method's role in approaching the unknown solution and the role of change in the initial values (small variations) for a first approximation to the solution. This gave meaning to differential equations by treating the derivative as the slope of the tangent line of the desired solution, which could be called the local linearization of the solution.

Therefore, the original contribution of this investigation lies in its use of *bounded variation*. The study of change and variation was insufficient to construct the notion of the uniqueness of the solution. Hence, and considering the problematics of studying the nature and epistemology of this knowledge, we proved that *bounded variation* contributed to the search for, or establishment of, conditions for the study of change and variation in the interpretation of the solution of differential equations, to obtain an adequate solution with a predictive goal – conditional (conjecture) – thus avoiding the diversity of possible solutions to the problem. A future study will focus on testing these ideas by designing a didactic intervention for university students. This will include designs appropriated from their profession, where the inverse nature of the problem is the centre of the design that will demonstrate the use of the existence and uniqueness of the solution of differential equations that characterise the phenomenon. That is another story, though.

AUTHORS' CONTRIBUTIONS STATEMENTS

RFS and RCU conceived the presented idea. RFS developed the theory. RFS and RCU adapted the methodology to this context, created the models, performed the activities, and collected the data. RFS analysed the data. All authors actively participated in the discussion of the results, reviewed and approved the work.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, RFS, upon reasonable request.

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