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Geometric Thinking: Reflections Manifested by Preservice Mathematics Teachers in van Hiele Model Studies

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ABSTRACT

Background: The study of geometric thinking in the preservice education of mathematics teachers is an emerging theme that can reverberate in the teaching of geometry in basic education. **Objectives:** To analyse reflections manifested by prospective mathematics teachers (PMTs), working with tasks supported by van Hiele theoretical model to develop geometric thinking. **Design**: The nature of this study is qualitative and interpretative. **Setting and participants:** Twenty-four PMTs members of a geometry teaching subject were investigated in a mathematics degree course at a public university in Paraná - Brazil. **Data collection and analysis**: The data was collected from the video-recorded training sessions, the written production of the PMTs promoted by the tasks and the registers kept on the field diary. The analysis focused on the reflections expressed by PMTs regarding the work with tasks involving geometric thinking, considering the levels of reflection proposed by Muir and Beswick (2007). **Results:** The results show descriptive, deliberate, and critical reflections, with different levels of incidence, associated with (I) the levels of thought proposed in the van Hiele model; (II) the teacher's role in classroom practice; and (III) the geometric concepts and properties of flat figures. **Conclusions:** The promotion of formative actions that privilege discussions and reflections on geometric thinking can allow PMTs to seek connections between knowledge of geometry, geometric thinking, and their future teaching practice.

Keywords: Preservice mathematics teachers' education; Geometry; Geometric thinking; van Hiele.

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Pensamento geométrico: reflexões manifestadas por futuros professores de matemática em estudos do modelo de van Hiele

RESUMO

Contexto: O estudo do pensamento geométrico na formação inicial de professores de Matemática é um tema emergente que pode reverberar no ensino de geometria na Educação Básica. **Objetivos:** Analisar reflexões manifestadas por futuros professores de Matemática (FPM), no trabalho com tarefas apoiadas no modelo teórico de van Hiele para desenvolver o pensamento geométrico. **Design**: A natureza do presente estudo é qualitativa, de cunho interpretativo. **Ambiente e participantes:** Foram investigados 24 FPM integrantes de uma disciplina de ensino de Geometria em um curso de licenciatura em Matemática de uma universidade pública do estado do Paraná - Brasil. **Coleta e análise de dados**: Os dados dessa investigação referem-se às sessões de formação registradas em vídeo, à produção escrita dos FPM promovidas pelas tarefas e aos registros em diário de campo. A análise incidiu sobre as reflexões manifestadas pelos FPM a respeito do trabalho com tarefas envolvendo pensamento geométrico, considerando os níveis de reflexão propostos por Muir e Beswick (2007). **Resultados:** Os resultados evidenciam reflexões descritivas, deliberadas e críticas, com diferentes níveis de incidência, associadas: (I) aos níveis de pensamento proposto no modelo de van Hiele; (II) ao papel do professor na prática em sala de aula; e (III) a conceitos geométricos e a propriedades de figuras planas. **Conclusões:** A promoção de ações formativas que privilegiam discussões e reflexões a respeito do pensamento geométrico pode oportunizar aos FPM a busca de conexões entre conhecimentos de geometria, de pensamento geométrico e de sua futura prática docente.

Palavras-chave: Formação Inicial de Professores de Matemática; Geometria; Pensamento Geométrico; van Hiele.

INTRODUCTION

Several studies suggest that in the preservice education of teachers who teach mathematics, tasks involving geometric thinking are developed so that prospective teachers can reflect and project the work with this theme in their future professional practice (Brunheira & Ponte, 2019; Erdogan, 2020; Livy & Downton, 2018).

Geometry is a system of representation used to visualise concepts, forms of reasoning, and spatial environments (Battista, 2007). Geometry teaching should contribute to developing visualisation and critical thinking skills, abilities to reason, argue, demonstrate, make logical assumptions and inferences, and reduce three-dimensional to two-dimensional objects. Its teaching should also help students realise that geometric ideas are valuable in representing and solving problems (Battista, 2007; NCTM, 2000).

Despite its relevance, geometry in Brazil is sometimes underworked or meaninglessly approached in basic education. Many teachers who teach this level of education do not feel prepared to teach geometry due to their precarious training in the content (Lorenzato, 1995; Nunes & Onuchic, 2019). According to Almouloud et al. (2004), in some cases, preservice education contributes little to prospective teachers reflecting on specific teaching and learning geometry issues. The authors suggest that the training space should encourage PMTs to understand what, how, why, and when to teach geometry.

Teachers report that during their education, geometry was reduced to recognising geometric figures, using meaningless formulas and procedures, and working with metric geometry without, for example, distinguishing figural aspects from geometric concepts, finally, without having experienced a geometry teaching that allowed them to think geometrically (Nacarato & Passos, 2003).

Livy and Downton (2018) argue that mathematics teachers' preservice education should include situations in which the prospective teachers not only develop their geometric thinking but also discuss pedagogical approaches that support the development of their students' geometric thinking. In this sense, some researchers (Brunheira & Ponte, 2019; Ferreira & Barbosa, 2013) highlight the importance of creating formative spaces capable of promoting interactions between the educator and the prospective mathematics teachers (PMTs), so that they can verbalise their reasoning, debate divergent ideas, build arguments, in other words, actively engage in the construction of geometric knowledge.

One of the most used theoretical models in research on geometric thinking in contexts of preservice teachers education who teach mathematics is that of van Hiele (Cybulski, 2022). In this model, geometry learning occurs through the evolution of the student's knowledge through five hierarchical levels of thought, each of which describes the thinking processes used in geometric contexts (van de Walle, 2009).

Thus, we emphasise the need to analyse reflections manifested by PMTs in working with tasks involving geometric thinking, supported by van Hiele theoretical model, in a geometry teaching subject in a mathematics degree course. It is especially relevant that teachers' preservice education focuses on promoting PMTs geometric thinking, in the case of van Hiele theoretical model, in determining thinking ability. Three main aspects underpin this theoretical model, which has levels of understanding. Each level has its characteristics, and the previous levels must be fully understood so students can achieve the next one (Knight, 2006). The study of this model guided part of the work of a geometry subject in PMTs' training, from which the present investigation takes shape.

THE VAN HIELE MODEL

The theoretical model proposed by the Dutch couple of mathematics educators, the van Hieles,[1] has provided insights into differences in geometric thinking and how these differences are established (van de Walle, 2009). The ontogenesis of individuals' geometric thinking consists of five hierarchical and consecutive levels: visualisation, analysis, informal deduction, deduction, and rigour (Alex & Mammen, 2018). These five levels of thought are characterised by the hierarchy established between them. They describe

> *how* we think and what kinds of geometric ideas we think about more than the amount of knowledge or information we have at each level. A significant difference from one level to the next is the *objects of thought* – about which we can *think* [operate] geometrically. (van de Walle, 2009, p. 439, emphasis in the original)

On the first level, visualisation, the objects of thought are the shapes and "what they look like" (van de Walle, 2009). At this level, figures are judged by their appearance and recognised by their different shapes, not by their properties. For example, a child can reproduce different forms if someone has already shown him/her such figures; however, they cannot establish relationships with the properties of these forms (van Hiele, 1984). Thus, the teacher will be able to explore the similarities and differences between them, aiming to use these ideas to create classes of forms (Van de Walle, 2009). The properties in these classes, such as parallel sides, right angles, and symmetries, can be included at this level, however, informally and observationally. Then, the product of thought, i.e., the ideas created at one level, become the focus or

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[1]Pierre van Hiele was a renowned researcher in the teaching of geometry who, along with his wife, Dina van Hiele-Geldof, investigated the development of geometric thinking, the first results of which began to be published in 1959.

object of thought of the next level, being at this level the classes or the cluster of similar forms (van de Walle, 2009).

At the second level, analysis, the objects of thought are the classes of forms, more than the individual forms (van de Walle, 2009). Figures are recognised by their properties, however, they are not yet ordered, so that, for example, “a square is not necessarily identified as a rectangle” (van Hiele, 1984, p. 245). Thus, the teacher can propose tasks in which the student is invited to think about, for example, what leads a geometric object to be classified as a rectangle and what other shapes can be grouped with that object, so that they have the same properties within a given class. In this way, ideas about an individual form can be generalised to all forms that align in the same class. The products of thought, at this level, are the properties of forms (van de Walle, 2009).

On the third level, formal deduction, the objects of thought are the properties of forms (van de Walle, 2009) that can be ordered and deduced from each other. Although students do not yet understand the intrinsic meaning of deduction (van Hiele, 1984), they can already follow and appreciate a logical argument under an intuitive character. However, they do not understand an appreciation of axiomatic structure in a formal deductive system (van de Walle, 2009). At this level, the products of thought are the relationships between the geometric properties of the objects (van de Walle, 2009).

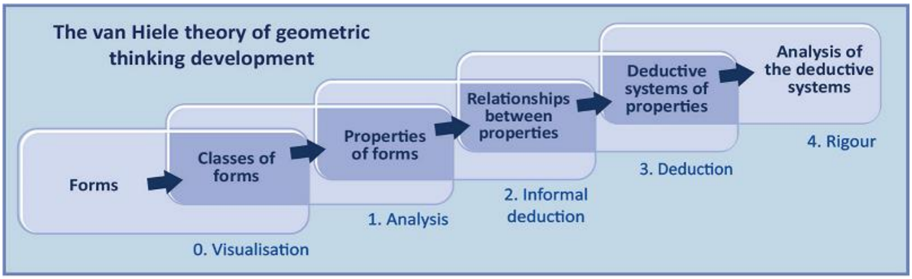
At the fourth level, deduction, the objects of thought are the relationships between the properties of geometric objects (van de Walle, 2009). Thought is centred on the meaning of deduction (van Hiele, 1984). At this level, students “can work with abstract sentences about geometric properties and draw conclusions based more on logic than intuition” (p. 443). The products of thought at this level are axiomatic deductive systems for geometry (van de Walle, 2009).

Finally, at the last level, rigour, the objects of thought are the axiomatic deductive systems for geometry (van de Walle, 2009). At this level, “figures are defined only by symbols linked by relations” (pp. 248-249), and the student – usually a specialist in mathematics in higher education – makes an appreciation of the distinctions and relationships between different axiomatic systems (van de Walle, 2009). Thus, the products of thought at this level are comparisons and confrontations between the different axiomatic systems of geometry.

The products of thought at each level become the objects of thought of the next level, i.e., the ideas created at one level become the focus or object of thought of the subsequent (van de Walle, 2009). This object-product relationship between the levels is illustrated in Figure 1.

Figure 1

Levels of thinking - Object-product relationship (van de Walle, 2009)



To van Hiele (1999), the development of the student's thinking depends more on the types of experiences offered than age or biological maturation. The author states that the instruction to promote the transition from one level to another goes through five phases[2], including sequences of tasks, which begin with the exploratory phase and allow the gradual construction of concepts. He also states that the teacher must be aware of this transition -which occurs continuously- because its instructions are decisive for the student to succeed in this process (van Hiele, 1984).

The model proposed by van Hiele identifies progressive learning because intuition, reasoning, and geometric language advance gradually and globally. It also understands that geometric knowledge implies previous experiences since it allows the student, as they undergo different experiences, to build mathematical ideas of learning (Mattos & Serrazina, 1996).

Thus, it is relevant to create education spaces in which PMTs can reflect on the work with tasks supported by van Hiele theoretical model.

[2]The phases described by van Hiele (1984, 1999) are: inquiry, directed orientation, explication, free orientation, and integration.

CONTEXT AND METHODOLOGICAL PROCEDURES

This qualitative research[3] (Bogdan & Biklen, 1994) was conducted during a geometry teaching subject[4] offered for the second year of a degree in mathematics at a public university in Paraná, Brazil. Twenty-four PMTs participated in this course, the teacher educator (TE) of the course and the first author of this article. The subject was organised in a virtual learning environment, the *Modular Object-Oriented Dynamic Learning Environment* (Moodle), in weekly synchronous classes via *Google Meet* at regular hours at night and with an average duration of 90 minutes[5] throughout 2021.

The first author of this article observed and analysed the actions developed in the subject and played an active role in the discussions with the research participants, factors that characterise this investigation as action research. In this type of investigation, the researcher, inserted in the research environment, can "observe it, understand it but, above all, change it in directions that allow the improvement of practices and greater freedom of actions and learning of the participants" (Fiorentini & Lorenzato, 2012, p. 112).

Table 1

Description of tasks proposed by the TE

Tasks	Description
Task 1: The proposition of the van Hiele test	Before studying this theory, the TE suggested that the PMTs perform a van Hiele test[6], consisting of 15 plane geometry questions. Then, to verify the PMT's geometric knowledge, she asked them to justify their answers.
Task 2 Theoretical study	The text "Thinking and Geometric Concepts" (Van de Walle, 2009) was studied in advance by the PMTs, discussed during class in small groups (four groups with five PMTs each and a group with four, here

[3]Approved by the Ethics Committee (Opinion: 5.001.063; CAAE: 50991921.1.0000.5231).
[4]The total workload foreseen for the subject was 120 hours.
[5]A lower workload was established in the 2021 school year, which remained in the emergency remote teaching modality, due to the Covid9 pandemic.
[6]Nasser & Santanna (1997, p. 85-87).

	represented by G1, G2, G3, G4, and G5) and, subsequently, with all participants in the subject.
Task 3 The proposition of a questionnaire approaching the text studied	The TE requested that the PMTs, gathered in the same groups as Task 2, answer a questionnaire about the characteristics and levels of thinking; the importance of the teacher understanding how these levels of thinking influence the learning process in geometry; and the evaluation in terms of the development of the student's geometric thinking.
Task 4: Task 3 Presentation and discussion	Each group presented its considerations regarding the answers given to the questionnaire (Task 3), which were followed by discussions with all participants of the discipline.
Task 5: Presentation and discussion of van Hiele test results	PMTs test results were presented by the TE, who discussed with them geometric concepts of flat figures involved in their answers, as well as answers given by students of basic education for this same test and possible pedagogical practices for teaching these concepts.

The general objective of the teaching plan of the component, prepared by the TE, is to promote reflections, discussions, and actions in the formation of PMTs on geometry teaching for basic education. The specific objective is to foster situations that promote the development of geometric thinking. Therefore, the TE proposed tasks (Table 1) to trigger PMT's reflections, considering the pursuit of these objectives.

Twenty-four PMTs participated in the present investigation. However, only 18 performed Task 1, although they participated in the other tasks. The data of this investigation refer to the training sessions recorded on video (*Google Meet*), the written production of the PMTs in the resolution of the tasks (Table 1), and the registers in the field diary. Intending to preserve the participants' anonymity, in the results, we used the acronyms PMT1 through PMT24 to represent each one of them.

In analysing the information, we identified patterns in the written production of the PMTs and the class discussions. Next, we examined those data in detail to identify aspects of geometric thinking contained in the reflections manifested by the PMTs during the development of the tasks.

We assume the levels of reflection proposed by Muir and Beswick (2007) as the lens of the analysis of the emerging reflections of the data in this investigation, because we agree that reflection is one of the supports for the (prospective) teachers' learning (Muir & Beswick, 2007). For the authors, the levels of reflection are technical description, deliberate reflection, and critical reflection. At the technical description level, the participant describes general accounts of classroom practice, often focusing on technical aspects, without weighing the value of experience. At the level of deliberate reflection, the participant identifies critical incidents[7] and justifies or explains the action or behaviour. Finally, at the critical reflection level, the participant goes beyond identifying critical incidents; he/she provides explanations to consider the perspectives of others and offer alternatives. Taking into account this reference, the results show descriptive, deliberate, and critical reflections with different levels of incidence, associated with: (I) the levels of thought proposed in the van Hiele model for the development of geometric thinking; (II) the role of the teacher in classroom practice for the development of geometric thinking; and (III) geometric concepts related to properties of flat figures.

REFLECTIONS MANIFESTED BY PMTs ON WORK WITH TASKS SUPPORTED BY VAN HIELE THEORETICAL CONTRIBUTIONS

In this section, we present the reflections manifested by PMTs in working with tasks supported by van Hiele theoretical contributions, associated with the levels of reflection proposed by Muir and Beswick (2007).

I Reflections related to the levels of thought proposed in the van Hiele model for the development of geometric thinking

In this subsection, we discuss the reflections manifested by the PMTs (Table 2) in the discussion of the text (Task 2) on the levels of thought for the development of geometric thinking according to the theoretical model proposed by van Hiele.

[7]To Muir and Beswick (2007), critical incidents are particular events involving teacher- or student-specific comments that seem to provide clear examples of some aspect of the student's practice or characteristic of thought.

Table 2

PMT Reflections on Geometric Thinking Levels

Levels of thinking	Evidence of reflections expressed by the PMTs
Level 0[8]	*Level 0 is related to the observation part. The student's perception that the figure is a square, a triangle. It depends much on the student's observation,* ***analysing the difference from one figure to another through observation*** *(G1).*
Level 1	*At level 1, the student continues to use the visualisation of the properties, but at this level,* ***he/she creates a sense of starting to classify the properties****. For example,* ***they can already analyse that a cube has six faces, and these faces are congruent*** *(G2).*
Level 2	*At level 2, informal deduction, the student begins to develop thinking in an almost formal way.* ***Students begin to think about the properties of geometric objects without the constraints of a particular object, beginning to develop the relationships between properties****. For example, if the four angles of a figure are straight, this implies that the shape is a rectangle. At this second level,* ***students can make these deductions informally about properties*** *(G3).*
Level 3	*At level 3,* ***students understand geometry as a deductive system****.* ***They can examine more than the properties of forms****. The thought, developed previously, allowed to establish relationships between properties. At this level of thinking,* ***students are capable of more logical than intuitive thoughts*** *(G4).*
Level 4	*Level 4, rigour, is the highest of this hierarchy. The focus is the axiomatic systems themselves and not just the deductions as in the previous levels and is usually a level of experts in*

[8]The numbering, used to order the thought levels proposed by van Hiele, was suggested by van de Walle (2009).

*mathematics. **I believe this level is the researcher's level, because she is studying Geometry in a doctorate** (G5).*

Note: In the discussion, the TE asked each group to present their reflections on one of the levels (e.g., G1 on Level 0; G2 on Level 1, etc.). The bold excerpts accentuate the types of geometric ideas that students can think (operate) geometrically at each level.

The PMTs denoted deliberative reflections by identifying central elements related to the levels of thought, and explained the types of geometric ideas that students can think (operate) geometrically at each level (van de Walle, 2009). The emphasis given by the PMT of what should be "expected" of the student at each level may have occurred because they are inserted in the context of preservice teacher education.

Discussions about the objects and thought products of each level were promoted through the tasks. For example, PMTs were asked to describe the first three levels of geometric thinking of van Hiele theory (Levels 0, 1, and 2); indicate in their descriptions the object and product of thought of each level; and realise how these ideas establish a progression from one level to the next. The following are the answers by G1, representative of the other groups, to Task 3 (questionnaire).

> *Level 0 – the object is the visualisation.*
>
> *Level 0 thought products are classes or clusters of forms that are "alike." Properties are included informally and observationally. **Students need to analyse whether particular cases can be generalised, activities of clusters of forms are proposed, preparing students for Level 1**.*
>
> *Level 1 – the object is the analysis.*
>
> *The products of thought are the properties of forms. Students will have contact with the properties of the figures, they will be able to **apply the ideas to an entire class of figures. At this level, critical thinking and reasoning will be developing**, **through this development, they will be being prepared for the level.***
>
> *Level 2 – the object is the informal deduction.*
>
> *The products of thought are the relationships between the properties of geometric objects. Students are encouraged to*

> ***elaborate and test hypotheses, logical arguments, and use informal language. Through these experiences, they will be prepared for the next level.***

The PMTs identified and described the object and the associated thought products at each level, providing evidence of reflections of a descriptive nature. In addition, they showed critical reflections when indicating alternatives for working with students (bold), which are actions capable of promoting the transition from one level of thought to the next. This reflective level is considered the highest level of reflection (Muir & Beswick, 2007), and can trigger the PMTs reflective process. Anchored in theory, they can express aspects that reveal a holistic perception of the process of teaching and learning geometry, namely geometric thinking.

During the discussions of Task 4, the PMTs also highlighted the relationship between object-product levels, present in van Hiele theory. This recognition is important for them to realise that "objects must be created at one level so that the relationships between these objects can become the focus of the next level" (van de Walle, 2009, p. 443). Recognising this relationship becomes essential since, for the development of geometric thinking to occur, according to this theoretical model, it is up to the teacher to be aware that their guidelines are decisive for the students to succeed in this process (van Hiele, 1984).

Also in Task 3, the PMTs were asked to describe the four characteristics of van Hiele thinking levels. To illustrate, we chose G4's answer, since, in general, the other groups pointed to the same characteristics.

> *Levels are sequential (...) Levels are not age-dependent in the sense of Piaget's stages of development. Geometric experience is the simple factor of greatest influence upon the advancement or development through the levels. When teaching or language is at a higher level than the students', there will be a lack of communication.*

The PMTs denoted descriptive reflections associated with general aspects and levels of thought, such as the movement from one level to the other is sequential, gradual and continuous. According to van Hiele (1984), to develop geometric thinking, the transition from one level to the next only happens if enough symbols are accumulated, leading to this new level, i.e., after so many concepts have been condensed in the symbols, these can be used as a guide to the next level.

II Reflections related to teachers' role in classroom practice for the development of geometric thinking

In this subsection, we present the reflections manifested by the PMTs related to teachers' role in classroom practice, especially regarding the development of geometric thinking. These reflections were evidenced at different times in the development of the tasks. For example, in Task 3, the PMTs were asked what they could do when students were at different levels of geometric thinking. G3 provided the following questionnaire response:

> *At first, we have to find out at what level each student is, through activities one must examine the students' speeches and through them and through constant observation, we can characterise their level of thought. Afterwards, appropriate activities should be applied to each level; we can even present activities that contemplate two levels of thinking; teamwork is also of great importance, due to the exchange of knowledge and dialogue between students. Thus, we seek for the student to reach a higher level of geometric thinking and also be able to develop the class in an integral way (G3).*

At the time they discussed the answers given to the questionnaire (Task 4), G5 unveiled the following reflection:

> *the teacher must understand the students' level to be able to intervene positively in the student's learning process, taking into account the knowledge already acquired so that later they can proceed to the next level understanding the **progressions of ideas and how they are constituted through the observation and classification of them** (G5).*

The PMTs highlighted significant aspects of their ideas about the teachers' role and teaching practices that favour the process of developing students' geometric thinking. In addition, they give evidence of deliberate reflections by identifying critical incidents such as the level of thought mobilised by them, the opportunity for everyone to progress in terms of thinking development at hierarchical levels, the organisation of content to meet the students' needs, students' prior knowledge, the use of a vocabulary appropriate to the context at each level, and learning through tasks that promote the evolution of the student from one level to another. However, when signalling the importance of these teaching practices as an alternative to working with geometric thinking in the classroom, they suggest reflections of a critical dimension, corroborating the issue proposed by the TE, which enhanced this type of reflection.

On the other hand, other discussions about teaching practices limited progressive learning in relation to the levels of thinking proposed by van Hiele. During the discussion of the text (Task 2), the PMTs mobilised reflections, such as:

> *If the teacher uses languages or understandings that are above the student's level or that have not yet been developed with them, this only* ***stimulates a mechanical learning.*** *(...) if the teacher chooses a student at level 0 or 1 and asks them about axioms, things that they do not know yet, instead of developing their thinking, they will only* ***be rote-learning and reproducing what they heard*** *(PMT9).*

Reflections on these teaching practices also appeared in the responses given to the questionnaire (Task 3). For example:

> *As, according to van Hiele, the learning of geometry occurs at hierarchical levels, so if they are given teachings that go beyond those already learned, the student* ***will not be able to correctly fix the concept*** *(...). Levels are sequential (...) when teaching or language is at a higher level than that of the student, there will be a* ***lack of communication*** *(G4).*

The PMTs highlighted the issue of working with tasks not consistent with the level of thinking presented by the student and how this can affect the development of geometric thinking, generating a lack of communication between the teacher and the students and, consequently, learning without assigning meanings to geometric concepts. This denotes reflections of a deliberate nature, which justify the teacher's actions and relate them to the students' activities, in the context of the development of geometric thinking, according to van Hiele model.

Evaluative practices were also the subject of discussions throughout the course. PMTs were asked how a teacher can evaluate students in terms of their overall geometric development or spatial sense. The following are the questionnaire responses (Task 3), provided by G1 and G4, representative of the other groups.

> *For the teacher to evaluate students' development, they must analyse the way they think about geometric shapes, how they*

> *understand and associate properties and concepts, the way to* ***apply the knowledge acquired in problem solving*** *(G4).*
>
> *Assuming van Hiele theory is correct, there may be students at different levels within a single class. Therefore, it is necessary to evaluate them so that it is possible to distinguish each student's level. In this case,* ***concrete materials, drawings and computational models can be used****, so, while the teacher applies the activity, he/she needs to be attentive and listen to the types of observations of his/her students. It is extremely important to know at what level each student is, because only then will it be possible to help him advance to the next level (G1).*

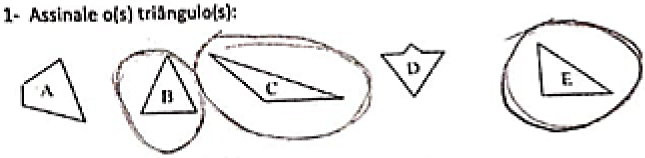
The PMTs pointed out that, when evaluating the students in terms of the development of geometric thinking, the teacher must observe how they understand the mathematical concepts built during classes and for this to occur, it is important to listen to them. The PMTs emphasised the importance of using a manipulative and visual resource as a tool for teaching geometry. These reflections are critical in nature because, in addition to identifying how to evaluate in terms of the development of geometric thinking, they proposed alternatives so that the action of evaluating serves as an instrument for the teacher to understand at which level each student is, thus enabling their decision-making. As van de Walle (2009) states, every teacher, when evaluating, must be able to perceive, throughout the worked year, some indication of development in the student's geometric thinking.

III Reflections related to geometric concepts regarding the geometric properties of plane figures

The discussions that took place during the development of Task 5 provided PMTs with a reflection on geometric concepts associated with properties of flat figures (triangles and quadrilaterals). We illustrate some answers and justifications given to the questions in Task 1, which were selected by the TE to discuss Task 5 (Figure 2).

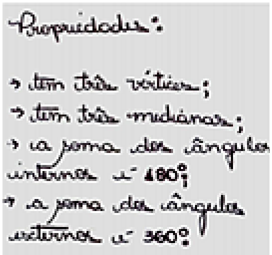
Figure 2

Answers and justifications provided to test question 1



Frequency of the answers: Seventeen PMTs scored B, C, and E; one scored B, C, D, and E.

Some PMTS' justifications:



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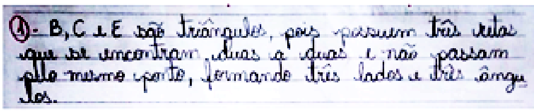

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In addition to the justifications presented in Figure 2, TE highlighted in her speech those that were most recurrent: *i) B, C, and E are triangles, as they have three vertices; ii) they are triangles because they have three angles; and iii) they are polygons formed by three sides.* She also commented that the PMT that marked alternative D registered that he was in doubt and incorrectly wrote that "if we divide the figure, we have two triangles". However, such a "division" could not be considered, as they would have to analyse the figure as a whole.

[9] Properties: it has three vertices; it has three medians; the sum of the interior angles is 180º; the sum of the three exterior angles is 360°.

[10] 1) They are triangles – B, C, E (they have 3 sides).

[11] 1) B, C and E are triangles, as they have three lines that meet two by two and do not pass through the same point, forming three sides and three angles.

We also highlight the following justification: *B, C, and E are triangles, as they have three faces (Discussion of the questionnaire-FMT12).* Based on this justification, a dialogue was established between the TE and the class.

TE:	*Does the triangle have faces?*
PMT7:	*No.*
TE:	*Why not?*
PMT7:	*Because it's not a spacial figure.*

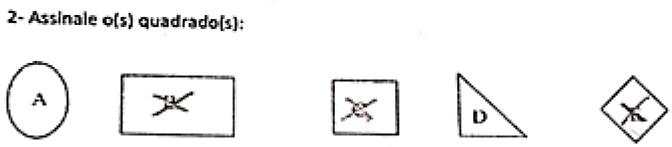
(Test Discussion – Task 5)

When answering the question raised by the TE, at first, PMT7 did not justify his answer; however, when provoked by the TE, he provided an explanation, thus denoting a reflection of a deliberative nature.

For the discussion of question 2 (Task 1), the TE selected the following answers and justifications (Figure 3):

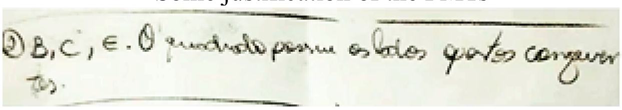
Figure 3

Answers and justifications provided to question 2



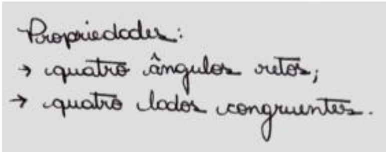
Frequency of responses:	Seven PMTs scored C; five scored B, C, and E; five scored C and E; and one scored B and C.

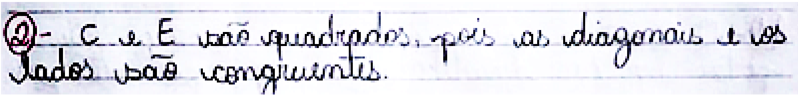
Some justification of the PMTs

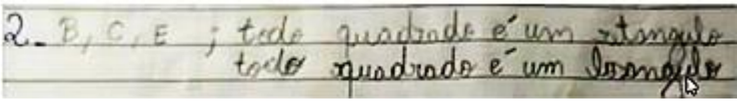


[12]

[12] 2) B, C, E. The square has congruent opposite sides.


Propriedades:
→ quatro ângulos retos;
→ quatro lados congruentes.
[13]


2- C e E são quadrados, pois as diagonais e os lados são congruentes.
[14]


2. B, C, E ; todo quadrado é um retângulo
todo quadrado é um losango
[15]

To provoke the discussion, TE invited the PMTs to interpret the answers (Figure 3), saying that they had been presented by students of basic education, not to subject those who erred to a situation of vulnerability.

TE: *What do you understand when the basic education students marked forms B, C, and E?*

PMT8: *I understand that students know little about the definition of a square.* ***They think to be a square is to have four sides. So any figure that has four sides is a square.***

TE: *And the student who scored only C, what can he know?*

PMT8: *He already knows that the opposite sides have to be equal (same measure) and that all sides have to be equal.*

TE: *What about the students who flagged C and E?*

PMT8: *They have a better understanding of a square.*

PMT7: *Teacher, I think that students who have marked B, C, and E do not necessarily know the*

[13] Properties: four right angles; four congruent sides.
[14] 2) C and E are squares because the diagonals and sides are congruent.
[15] 2) B, C, E – every square is a rectangle; every square is a rhombus.

	definition of a square. ***They may have thought it is a rectangle, if we "cut" it in half, it becomes two squares****.*
TE:	*But look at the statement, it is very consistent, it asks to mark the squares, it does not ask to make a section. And any rectangle that's split in half, does it become two squares?*
PMT7:	*No.*
PMT8:	***Figure E looks like a rhombus, but we cannot say that the four sides are equal looking only at the figure because every square is a rhombus, but not every rhombus is a square.*** *So whoever marked this figure assumed that the rhombus is a square.*
TE:	*It is really not possible to say, because that figure is a little suspicious, there may have been some deformation when cutting at the time of assembling the test. But as you said, the diamond is a figure with four sides with the same measurement. So if the student knows this definition, he may have thought about it, but he may have looked only at the appearance of the figure. This is good because sometimes students only think of squares when presented as the letter C, but if we rotate the figure, it does not lose its properties. And usually, in classrooms and textbooks, the squares are presented as in the letter C; hardly they present as in letter E.*

(Test Discussion – Task 5)

The reflections manifested by the PMTs result from the attempt to understand ideas and reasonings manifested by students of basic education, which worked as a stimulus and deepening of their geometric knowledge related to the properties of flat figures, considering that PMTs, when signing figure B (rectangle) as a square, showed difficulties with the recognition of properties of flat figures.

PMT8 expressed deliberative reflections, stating that, as students may know little about the definition of the square, they consider that every figure with four sides is a square.

On the other hand, PMT7 gives a possible justification for the answer of the students who indicated C and E, which shows a critical reflection. However, his justification demonstrates his fragility in relation to this geometry content. This experience can lead PMTs to learn geometry and how to teach it**.**

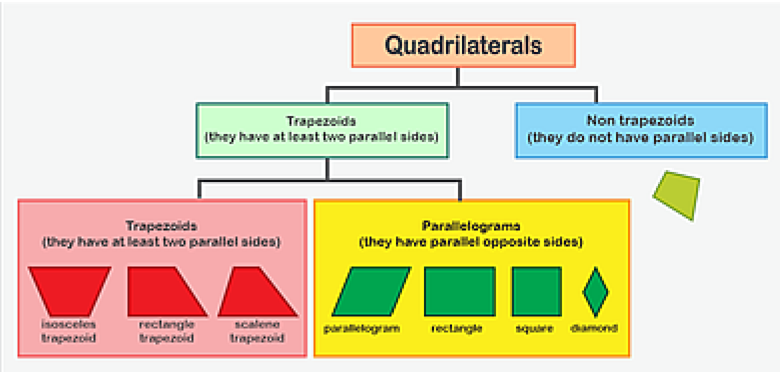
To explore the properties of flat figures, TE provoked a discussion to systematise the properties of quadrilaterals and group them into classes.

> *TE:* *Does the square have opposite and congruent sides? What about rectangles? What about the rhombuses?*
>
> *PMT8:* *Not always. Thinking about the definition of the square, it is also considered a parallelogram.*
>
> *TE:* *And parallelograms have opposite sides that are parallel and congruent. So, they are trapezes?*
>
> *(Test Discussion – Task 5)*

The TE's questioning led the PMTs to critically reflect on the systematisation of quadrilateral properties, which may favour the deepening of understandings and clarify misconceptions about the inclusion of quadrilateral classes. It is noteworthy that when the TE raised the question about the trapezoid, the PMT did not answer the question, which led him to present the properties of the trapezoid, so he concluded that all parallelograms are trapezoids (Figure 4).

Figure 4.

Classes of quadrilaterals presented by the TE



Parallelograms:

- *They have opposite congruent sides.*
- *They have congruent opposite angles.*
- *They have additional adjacent angles.*
- *The diagonal of a parallelogram intersects at its midpoints.*

From the systematisation carried out in the development of Task 5, we evidenced that the reflections arising from this discussion provided evidence of the construction of geometric concepts related to flat figures, such as: definition and properties of triangles, definition and properties of quadrilaterals, and the inclusion of classes of notable quadrilaterals.

DISCUSSION OF THE RESULTS

The reflections emerging in the development of the tasks regarding geometric thinking were mostly identified as deliberate and critical, which reveals that formative spaces, such as the one promoted in this discipline, may be promising for future teachers to reflect *on* the practice (Muir & Beswick, 2007) in the teaching of geometry. The PMTs, when experiencing situations of pedagogical practice promoted in the work with the tasks, could reflect on what, how, why, and when to teach geometry.

Considering that geometry promotes logical thinking and mathematical understanding, mathematics teachers play a crucial role in the teaching and learning process of this theme (van Hiele, 1999). Therefore, knowledge from a theoretical perspective, to develop geometric thinking – such as van Hiele's, for example – in preservice education, can not only expand PMTs' geometric thinking but also the search for ways to support this type of thinking of their future students (Livy & Downton, 2018; Nacarato & Passos, 2003).

The reflections about the levels of thought suggest they recognise the proposed objects and products of thought of each level and identify general characteristics of the theoretical model studied. Such reflections were, for the most part, deliberate. Supported by theoretical studies (Task 2), the PMTs identified central elements of each level proposed in the van Hiele model and explained how a student could think/operate at each level.

The reflections generated from the discussion about the role of the teacher in classroom practice were mostly deliberate and critical. The practical experience of analysing students' misconceptions, ideas, records and strategies, as well as their own, as proposed in Task 5, allowed PMTs to understand possible difficulties students face in developing geometric thinking. By analysing the answers presented from a teacher's perspective, they could observe different approaches and strategies, designing different ways to provide appropriate instructions to trigger the transition between their students' levels of thinking (Lee & Lee, 2020). The discussions promoted about pedagogical content knowledge, in this case, knowledge of geometry content, corroborate the idea that van Hiele model is an important resource, as it provides a rich basis for the (prospective) teacher's understanding of geometry and how students learn it (Alex & Mammen, 2018; Erdogan, 2020).

In this sense, PMTs highlighted relevant aspects, such as teaching practices that favour or disfavour the process of developing students' geometric thinking and how to evaluate this way of thinking. Thus, by identifying the relevance and implications of these teaching practices when working with geometry in the classroom and offering alternatives to support the development of students' geometric thinking in future professional practice, PMTs showed reflections of critical dimension. We argue that the development of students' geometric thinking depends on the types of experiences offered to them, and it is up to the teacher to recognise the level of thinking of the students to propose appropriate tasks and give them the opportunity for the gradual construction of geometric concepts (van Hiele, 1999).

Regarding the construction of geometric concepts related to flat figures, we identified, for the most part, deliberative reflections, which demonstrated that the PMTs mobilised geometric knowledge regarding the definitions and properties of triangles, quadrilaterals, and the inclusion of notable quadrilateral classes. Thus, intentionally privileging specific knowledge of geometry, such as quadrangles, provided PMTs with a broader knowledge of this content. Often, the absence or infrequency of teaching geometry in basic education is related to the weaknesses of mathematics teachers in relation, for example, to deductive reasoning, to the misunderstanding of the classification process of quadrilaterals (Brunheira & Ponte, 2019; Costa & Santos; 2016, Fujita, 2012). Such weaknesses signal the importance of training processes and research that develop strategies to mitigate them, supporting the PMTs in the development of their geometric thinking (Brunheira & Ponte, 2019).

The execution of the tasks and the dynamics established by the TE in the training process enabled the PMTs to recognise and assign meanings to geometric concepts and properties, in addition to other actions that indicate the development of geometric thinking. Table 3 illustrates a synthesis of the geometric thinking aspects identified, based on the reflections manifested by PMTs in this formative process associated with: (I) the levels of thinking proposed in the van Hiele model; (II) the teacher's role in the classroom practice; and (III) geometric concepts and properties of flat figures.

Table 3

Aspects of geometric thinking identified from reflections

Associated reflections:	Aspects of geometric thinking
The levels of thought proposed in van Hiele model	• Analyse differences from one geometric shape to another through visualisation. • Classify properties of geometric shapes, in particular. • Establish relationships between different geometric objects to deduce properties informally. • Perform generalisations and construction of classes of geometric objects with common characteristics. • Work with abstract sentences about geometric properties. • Understand the complexity of phenomena and make inferences about them. • Recognise geometric objects through deductive processes, mobilising properties of these objects that now compose the abstract world.
The teacher's role in classroom practice	• Provide evidence that mechanical reproduction does not enhance the development of geometric thinking. • Observe whether students assign meanings to geometric concepts to mobilise them coherently in problem solving.

Geometric concepts related to the properties of flat figures	• Recognise constituent elements of geometric shapes and properties • Establish differences between elements of plane and spatial figures • Define a geometric object, in addition to just observing its characteristics. • Include quadrilaterals in classes according to their properties.

Such aspects may constitute a way of understanding geometry and its teaching. The knowledge manifested by PMTs in problem-solving can be helpful to understand phenomena of the physical world and different areas of knowledge, from the sensory exploration of objects present around it to the recognition of geometry as the lenses for understanding objects that make up the theoretical world (Costa, 2020).

CONCLUDING

The analyses and discussions manifested by the PMTs allow us to conclude that the moments of socialisation in classes – even with the difficulties met due to the pandemic context – were enriched by experiences provided by the tasks and the dynamics established by the TE These tasks and discussions enabled reflections for more critical and broad teacher education on theoretical knowledge about geometric thinking, classroom practices, and the construction of geometric concepts.

The different levels of reflection on which the analytical framework was based showed promise to clarify different modes – in a hierarchical and interrelated structure – by which reflection, based on working with well-defined tasks, influences and offers conditions to foster PMTs' learning about geometric thinking.

The emerging reflections demonstrated that, when operating at the highest level of critical reflection, the PMT may be able to envision possibilities for future practices for geometry teaching to associate theory with practice. During the preservice education process, this is contemplated a few times. Therefore, actions such as those triggered in this training process can contribute to minimising the dichotomy between the prospective teachers' difficulties in learning geometric contents and learning to teach geometry.

By interacting with the PMTs and promoting their interaction with each other, the TE allowed them to verbalise their reasoning, debate divergent ideas, investigate properties, and construct/systematise geometric concepts with meanings.

Formative actions that allow the prospective teacher to go through different reflective levels can result in a learning process that allows the construction and development of the geometric thinking necessary for their future professional practice. Moreover, further investigations, incidents in the observation and promotion of these training actions, supported by other theoretical models, may offer necessary elements to clarify other aspects of geometric thinking.

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AUTHOR CONTRIBUTION STATEMENT

A.F.M.V. e M.C.C.T.C. conceberam a ideia apresentada, desenvolveram a teoria e adaptaram a metodologia. A.F.M.V. coletou os dados. As autoras analisaram os dados, discutiram os resultados e contribuíram para a versão final do manuscrito.

DATA AVAILABILITY STATEMENT

The data supporting this article are under the custody of A.F.M.V. and may be made available upon request from other interested parties for five years.

REFERENCES

Alex, J. K. & Mammen, K. J. (2018). Students' understanding of geometry terminology through the lens of van Hiele theory. *Pythagoras, 39*(1), 1-8. https://doi.org/10.4102/pythagoras.v39i1.376

Almouloud, S. A., Manrique, A. L., Silva, M. J. F. D., & Campos, T. M. M. (2004). A geometria no ensino fundamental: reflexões sobre uma experiência de formação envolvendo professores e alunos. *Revista Brasileira de Educação, 27*, 94-108. https://doi.org/10.1590/S1413-24782004000300007

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 843-908). NCTM/Information Age Publishing.

Bogdan, R. & Biklen, S. K. (1994). *Investigação qualitativa em Educação: uma introdução às teorias e aos métodos*. Porto.

Brunheira, L. & da Ponte, J. P. (2019). From the classification of quadrilaterals to the classification of prisms: An experiment with prospective teachers. *The Journal of Mathematical Behavior*, *53*, 65-80. https://doi.org/10.1016/j.jmathb.2018.06.004

Costa, A. P. da. (2020). O pensamento geométrico em foco: construindo uma definição. *Revista Eletrônica Científica Ensino Interdisciplinar*, *6*(16). https://doi.org/10.36556/eol.v0i22.253 .

Costa, A. P. da & dos Santos, M. C. (2016). O pensamento geométrico de professores de matemática do ensino básico: um estudo sobre os quadriláteros notáveis. *Educação Online*, *1*(22), 1-19. http://doi.org/10.21920/recei720206167794 .

Cybulski, F. C. (2022). *Geometria na formação inicial de professores que ensinam matemática: indicativos de dissertações e teses brasileiras*. [Mestrado em Ensino de Ciências e Educação Matemática]. Universidade Estadual de Londrina.

Erdogan, F. (2020). Prospective middle school mathematics teachers' problem posing abilities in context of Van Hiele Levels of Geometric Thinking. *International Online Journal of Educational Sciences*, *12*(2). https://doi.org/10.15345/iojes.2020.02.009 .

Ferreira, A. C., & Barbosa, C. P. (2013). Saberes profissionais e pensamento geométrico: o caso de uma professora dos anos iniciais do Ensino Fundamental. *Acta Scientiae, 15*(1), 93-112.

Fiorentini, D., & Lorenzato, S. (2012). *Investigação em educação matemática: percursos teóricos e metodológicos* (3. ed., Coleção formação de professores). Autores Associados.

Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *The Journal of Mathematical Behavior*, *31*(1), 60-72. https://doi.org/10.1016/j.jmathb.2011.08.003

Knight, K. C. (2006). *An investigation into the change in the Van Hiele levels of understanding geometry of pre-service elementary and secondary mathematics teachers.* Unpublished doctoral dissertation, The University of Maine, Orono.

Lee, M. Y., & Lee, J. E. (2020). Spotlight on area models: Pre-service teachers' ability to link fractions and geometric measurement. *International Journal of Science and Mathematics Education*, *19*(5), 1079-1102. https://doi.org/10.1007/s10763-020-10098-2 .

Livy, S., & Downton, A. (2018). Exploring experiences for assisting primary pre-service teachers to extend their knowledge of student strategies and reasoning. *The Journal of Mathematical Behavior, 51*, 150-160. https://doi.org/10.1016/j.jmathb.2017.11.004 .

Lorenzato, S. (1995). Porque não ensinar Geometria? *A Educação Matemática em Revista*, *III*(4), 3-13.

Mattos, J. M., & Serrazina, M. D. L. (1996). Didáctica da matemática. *Universidade Aberta*, 191-212.

Muir,T., & Beswick, K. (2007). Stimulating reflection on practice: Using the supportive classroom reflection process. *Mathematics Teacher Education and Development*, *8*, 74-93.

Nacarato, A. M., & Passos, C. L. B. (2003). *A geometria nas séries iniciais*: uma análise sob a perspectiva da prática pedagógica e da formação de professores. EdUFSCar.

Nasser, L., & Santanna, N. P. (1997). *Geometria segundo a teoria de Van Hiele*. UFRJ, Projeto Fundão.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* National Council of Teachers of Mathematics.

Nunes C. B., & Onuchic L. R. (2019). O uso das transformações geométricas através da resolução de problemas na formação de futuros professores de matemática**.** *Interfaces da Educ.*, *10*(30), 30-56. https://doi.org/10.26514/inter.v10i30.3565 .

van de Walle, J. A. (2009). *Matemática no Ensino Fundamental: Formação de Professores e Aplicação em Sala de Aula*. Penso.

van Hiele, P. M. A. (1984). child's thought and geometry. In D. Fuys, D. Geddes, & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and P. M. van Hiele* (pp. 243-252). Brooklyn College.

van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. *Teaching children mathematics, 5*(6), 310-316. https://doi.org/10.5951/TCM.5.6.0310 .