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Straight Lines and Angles that Move, Students' Ideas that Touch and Add

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ABSTRACT

Background: Visualisation is a crucial skill for several everyday life activities and conceptual construction and development. **Objectives:** As it represents a substantial part of Euclidean geometry, this article addresses: the study and analysis of the relationships between angles formed from two parallel lines intersected by a transversal and the role of visualisation in the construction and development of geometric concepts imbricated in the mathematical relationships underlying the theme. **Design:** The methodological approach adopted was Design Experiments. **Settings and participants**: A teaching situation is presented in which students from the 8th grade of elementary school at a state public school in Rio de Janeiro, Brazil, performed manipulations on the smartphone screen while carrying out an activity. **Data collection and analysis:** To analyse this activity, we selected events from a video obtained from the smartphone used by the students. **Results:** Through interactions and handlings, students visualised, analysed, and built the concepts of corresponding angles and collateral angles by studying a geometric object on the smartphone screen through the GeoGebra application. **Conclusions:** The geometric work mediated by mobile devices with touch screens enables the curricular reorganisation of Geometry and the articulation and the breaking of the hierarchy of concepts.

Keywords: Parallel and transversal lines; Angles between lines; Conceptualisation; Visualisation; Dynamic geometry; Elementary school.

Retas e ângulos que se movimentam, ideias discentes que tocam e somam

RESUMO

Contexto: A visualização é uma habilidade indispensável para uma variedade de atividades da vida cotidiana e para a construção e o desenvolvimento conceitual.

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Objetivos: Por representar uma parte substancial da Geometria Euclidiana, abordam-se neste texto: o estudo e a análise das relações entre ângulos formados a partir de duas retas paralelas intersectadas por uma transversal; e o papel da visualização na construção e no desenvolvimento dos conceitos geométricos imbricados nas relações matemáticas subjacentes ao tema. **Design**: A abordagem metodológica adotada para o desenvolvimento do estudo foi o *Design Experiments*. **Ambiente e participantes**: Apresenta-se uma situação de ensino na qual estudantes do 8.º ano do Ensino Fundamental de uma escola pública estadual do Rio de Janeiro, Brasil, realizaram manipulações na tela do *smartphone* durante a realização de uma atividade. **Coleta e análise de dados**: Para análise dessa atividade selecionamos eventos de um vídeo obtidos a partir do *smartphone* utilizado pelos estudantes. **Resultados**: Por meio das interações e manipulações os discentes visualizaram, analisaram e construíram os conceitos de ângulos correspondentes e ângulos colaterais, mediante o estudo de um objeto geométrico na tela do *smartphone* por meio do aplicativo GeoGebra. **Conclusões**: O trabalho geométrico mediado por dispositivos móveis com toques em telas possibilita a reorganização curricular da Geometria, a articulação e a quebra de hierarquia de conceitos.

Palavras-chave: retas paralelas e transversais; ângulos entre retas; visualização; conceituação; geometria dinâmica; Ensino Fundamental.

INTRODUCTION

Mobile digital technologies - smartphones and tablets - enable other ways of thinking about teaching and geometric learning and allow for qualitative changes of a didactic nature with new teaching methodologies in the curricular organisation and cognitive dimension (Bairral & Henrique, 2021). Bringing reflections on didactic and cognitive possibilities of innovation, in this article, we approach the role of visualisation for conceptual construction and development of processes enhanced by dynamic geometry environments (DGE) on mobile touch-screen devices (MTSD), particularly the GeoGebra application in smartphones.

We discuss here some issues related to teaching and learning geometry. We present some singularities and potentialities of the DGE, and we try to provoke some questions, such as: Should the learning of Euclidean geometry in DGE continue to follow the hierarchical way in which it was organised for conventional resources? How is the construction and analysis of a geometric object mediated by clicks on the mouse different from the mediation of touches on the screen?

By allowing the approach of a range of themes related to Euclidean geometry, such as the study of triangles and quadrilaterals, we focused on analysing the relationships between angles formed from two parallel lines intersected by a transversal. To highlight some of our results, we illustrate a teaching situation in which two students from the 8th grade of elementary school worked on the analysis of the relationships between corresponding angles and collateral angles through the manipulation of the touchscreen mediated by the GeoGebra app in smartphones. Beyond the touches, we infer how the students appropriated them to have insights, visualise, and formulate their conjectures when repositioning angles and lines of geometric objects towards the conceptual construction. The research considers visualisation and conceptualisation as intertwined cognitive processes since the development of the ability to visualise composes the construction and conceptual development.

VISUALISATION AND GEOMETRIC CONCEPTUALISATION

In a broader context, visualisation originates from the Latin *visualis*, which is relative to sight, and we may understand it as the act or effect of seeing. The word "see", in this case, assumes not only the objects that are before the eyes, such as a printed book or a graphic representation on the smartphone screen but the mental images that we manage to form and manipulate – and it is towards this direction that our reflections converge.

The term "visualisation" in the geometric context has different conceptions. However, regardless of the adopted perspective, visualisation is seen as the ability to manipulate visual images[1]. In the same direction, Zimmermann and Cunningham (1991) point out the multifaceted nature of visualisation, as it has historical, philosophical, pedagogical, and technological perspectives.

Zimmermann and Cunningham (1991) expand the definition of mathematical visualisation. For the authors, it is not just manipulating a mental image, as it is a process that involves the representation of a concept, without or with the aid of technology, for understanding and mathematical discoveries. The authors also argue that visualisation implies a type of intuition that gives meaning to understanding, as it directs the construction of creative ideas and serves as a guide for problem solving. To get that kind of intuition, visualisation must relate to all of the mathematics. Therefore, "visual thinking and graphic

[1] Presmeg (1986, p. 46) defines visual image as "a mental schema that describes visual or spatial information".

representations must be linked to other modes of mathematical thinking and other forms of representation" (Zimmermann & Cunningham, 1991, p. 4).

Similarly, Leivas (2009) defines visualisation as a process of forming mental images, which implies constructing and communicating a mathematical concept to support the resolution of analytical or geometric problems. The author also adds the intuitive character to this process, considering that intuition is related to constructing a mathematical concept based on concrete experiences and the subject's analysis of the object.

Furthermore, Nacarato and Passos (2003) understand representation as an essential element for visualisation. They claim that visualisation and representation are two interconnected entities and play a fundamental role in the development of geometric concepts. They define visualisation as the ability to think in terms of a mental image, a mental representation of an object or relationship. Representation in this context takes on a double character: it is the evocation of an image or its presentation, which, as the authors explain, occurs in different ways, i.e., graphic presentations, such as a drawing on paper or a figure on a computer screen; or the language itself, and express the subjects' strategies in their geometric ideas.

In tune, Hershkowitz (1994) clarifies that visualisation plays a complex role in the process of conceptual development and acts in two directions. Thus, it is impossible to form the image of a concept and its class of figures without visualising its elements. But, on the other hand, if the visual elements are restricted and limited, they can impoverish the conceptual image.

When it comes to visualisation, there are nuances like visual reasoning, and visual thinking; and elements such as imagination, intuition, visual perception, and representation, which are related in the elaboration of the visualisation process. As stated by Gutierrez (1991), this process has as its fundamental entity the mental representations we make of physical objects, relationships, and concepts, among others.

Although some authors present different perspectives regarding the definition of visualisation, they complement each other and point out the role of visualisation in conceptual construction and development. Generally, studies that show visualisation leave conceptualisation at a different level and vice versa. Our intention here is to show that these processes are related and, to a certain extent, are intertwined, since the development of the ability to visualise composes construction and conceptual development.

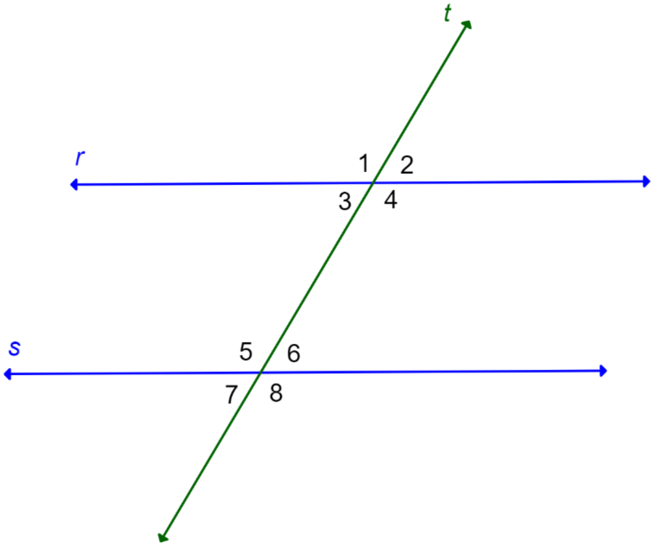
Concepts are the bridge between the mind and the world (Rosch, 1999). Still, according to Rosch, concepts are not isolated entities, and their categorisations only exist in concrete complex situations. Conceptual construction and development occur when a subject, based on their experiences, manages to elaborate a mental image and visualise the elements that cover not only the particular case but a whole class of objects (Almeida & Lomônaco, 2018; Ferreira, 1963; Fischbein, 1993; Medin, 1989). In this context, mathematical concepts are derived from their definition, as it establishes a cut between the instances that are examples of that concept and those that are not (Hershkowitz, 1994), i.e., the definition and the concept are related, the first being the limits of the second. Furthermore, the conceptual scope is contextual (Oliveira & Oliveira, 1999), metaphorically rich (Henrique & Bairral, 2020; Lakoff & Johnson, 2001), and allows the subject to develop their complex analogical and inferential skills (Wolf, 2019).

In geometry, another important element in articulating the processes that comprise visualising and conceptualising is the geometric object. We turn to Fischbein (1993), who discusses a study that aims to understand the role of the concept and the image in the composition of the geometric object. Fischbein maintains that it is necessary to consider the definition, the image and the figural concept as categories of a geometric object. It establishes a figural concept as a construction treated by mathematical reasoning in the geometry domain, controlled and manipulated by logical principles of an axiomatic system.

As Fischbein (1993) clarifies, the geometric object depends on a conceptual and a figural nature, and the balance between these two components allows the exact notion of the object. To the author, from the conceptual nature derives the general idea that expresses the class of objects, while the figural nature, the mental image, is the representation of an object or phenomenon. To better understand this relationship, it is important to highlight that the mathematical objects, such as points, lines, and parallel lines, are ideal models of mental entities, in which only in a conceptual sense is it possible to consider the perfection of these objects. They are general representations; they do not exist in the real world, they are abstractions that belong to the domain of concepts, adds the author. Let us consider the following postulate presented by Alexander and Koeberlein (2013, p. 75): "if two parallel lines intersect by a transversal, then the corresponding angles are congruent". This geometric object is constructed by abstract entities, and the validity of the relation (corresponding congruent angles), proved through logical deductions, can be explored through the DGE and depends on a mental manipulation for the conceptual construction. Let us look at Figure 1.

Figure 1

Corresponding angles formed from two parallel lines and a transversal line.



Imagine that you can move the line *r* so that it can be superimposed on *s*; in this way, we will see that the vertices of angles 1 and 5, for example, will coincide. As *r* and *s* are parallel lines, we can assume that the rays (the sides) of the angles will also coincide so that the validity of the relation will be verified. Using this same reasoning, it is possible to establish other manipulations, to prove the validity of the related theorems of this postulate (alternate angles and collateral angles). In mental manipulation, two types of images are involved: one of the geometric objects formed by two parallel lines and a transversal line and the other of the operation, manipulated from the displacement of one of the parallel lines, to verify the mathematical relationship.

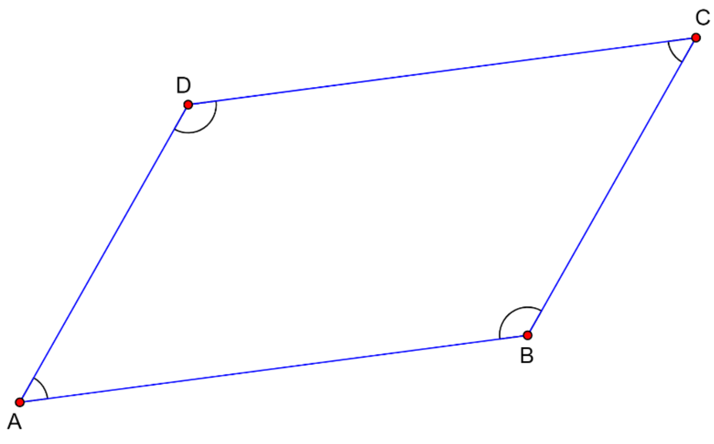
Fischbein argues that concepts do not move; it is impossible to displace them, nor do these objects exist in the real world, as they are representations in which there is an interrelationship between the conceptual and the figural. Thus, as the author claims,

> The objects of investigation and manipulation of geometric reasoning are then mental entities, called figural concepts, which reflect spatial properties (shape, position, magnitude) and, at the same time, have conceptual qualities, such as ideality, abstraction, generality, and perfection. (Fischbein, 1993, p. 143)

On the other hand, this process requires the ability to manipulate a mental entity, the visualisation. Corroborating these findings, it is possible to infer that construction and conceptual development are processes related to visualisation based on object analysis, which has two components: one figural and the other conceptual. Let us see an example that helps to clarify the statement: consider the quadrilateral ABCD, whose opposite sides are parallel (Figure 2). What is the relationship between ∠DAB and ∠BCD?

Figure 2

Exemplification of visualisation in the analysis of a geometric object.



Since all sides are parallel, we can deduce that ∠DAB and the exterior angle adjacent to the ∠ABC are congruent because they correspond. This process can be visualised by displacing the segment DA on the support lines that contain DC and AB. An immediate conclusion is that ∠DAB and ∠ABC are internal collaterals. From this observation, it follows that ∠DAB and ∠ABC are supplementary, and, in conclusion, ∠ DAB and ∠BCD are congruent.

The proposal of a problem involving the design of a geometric object starts from theoretical questions linked to the object, but the solution represents assumptions, as it is a particular case, a drawing. Laborde and Capponi (1994) distinguish geometric design from a geometric object. To the authors, the passage from the first to the second depends on the subject's previous experiences, the context, and the formed meaning. The geometric figure refers to the relationship between a geometric object and all its possible representations (Laborde, 1998). In this way, we can define the design of a geometric object as a particular case whose representation is associated with the subject's concept of the theory taken as a reference. When we work in DGE, we build a figure, not a drawing.

Furthermore, we perceive that the study of geometry plays an important role in cognitive (logical-mathematical) and social development. It enhances imagination, intuition and visualisation, contributing to conceptual construction and development.

The present research addresses visualisation as an essential cognitive process fed back by dynamic representations and by concepts and properties emerging on the screen of an DGE. Therefore, the concepts are constructed and given new meanings throughout the interactive process established between students, teachers and the proposed tasks.

ANGLES BETWEEN STRAIGHT LINES, DGE, AND MTSD

We take advantage of the importance of geometry and highlight visualisation as a necessary skill for cognitive development and some theoretical, didactic, and pedagogical implications intertwined with construction processes and conceptual development. In our investigation, as it represents a substantial part of Euclidean geometry, we focused on the study and analysis of the relationships between angles formed from two parallel lines intersected by a transversal.

Euclidean geometry follows a structure composed of primitive entities, such as a point, a line, and a plane; axioms or postulates, self-establishing truths; and definitions. By combining these elements and using a logical reasoning sequence, theorems are constructed (Kaleff, 1994). However, to learn geometry, is it necessary to follow this hierarchy?

Starting from the study of parallel lines cut by a transversal, one can approach theorems related to the study of triangles (exterior angle theorem[2] and the sum of the interior angles[3]) and the relationships existing in some quadrilaterals, such as the relationship between opposite angles or adjacent angles of a parallelogram[4]; one can also use properties as a foundation to address other concepts, such as Thales' theorem[5] and the deepening of the theme itself, for example, the relationship between the angles formed by polygonal lines between two parallel lines.

However, we do not believe following this sequence is necessary. It is possible to (re)visit other concepts: intersecting lines, angles opposite by vertex[6], supplementary angles[7], bisector,[8] and the very idea of angle[9], for example.

Regarding the approach to this theme, it is worth noting that, traditionally, textbooks emphasise the presentation of nomenclatures and the proposal of questions whose focus is much more on solving equations than on analysing the relationships between angles, despite the overvaluation of static images, without a proposal of dynamic resources.

The static way geometry is approached can have didactic and epistemological implications. For example, Fischbein (1993) highlights the

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[2] The exterior angle of a triangle is equal to the sum of the two other non-adjacent interior angles.

[3] It results in two right angles.

[4] In every parallelogram the opposite angles have the same measure, and the sum of the adjacent angles gives two right angles.

[5] According to Thales' Theorem, two transversal lines in a bundle of parallel lines form proportional segments. Thus, considering parallel lines, the ratio of any two segments of one line is equal to the ratio of the corresponding segments of the other.

[6] They are angles that have the same measure.

[7] Angles whose sum is two right angles.

[8] Ray internal to the angle, with origin at the vertex and which divides it into two angles of the same measure.

[9] A geometric object formed by the union of two (non-collinear) rays with the same origin.

importance of exploring aspects inherent to properties and definitions instead of sticking only to the figure when performing tasks involving geometric problems. In this way, it seems more acceptable that geometry teaching is guided by manipulation, exploration and, therefore, discovery based on the formulation of conjectures.

Digital technologies expand the possibilities of working with geometry, as they place experimental power in the hands of students when developing an activity. Thus, the core may lie in discovering or understanding relationships, properties, and theorems. Such arguments direct us to the relevance of geometric learning mediated by DGE, which, in our view, breaks with Euclidean logic by enabling a decentralised approach to this hierarchy and even allowing the construction of new concepts.

Just as the investigation of Euclid's fifth postulate [10] enabled the discovery of other geometries, the investigations in an DGE, initially on computers and recently on MTSD, as smartphones and tablets, use a geometry that enables the construction of new concepts from the analysis of old geometric entities. As Bairral (2019) alleges, different types of devices generate different insights for our learning. The interaction with them enables the development or creation of concepts. In this regard, the author clarifies that it is possible to create a mathematical concept. As an example, he uses the case of operations with angle measures, such as degrees, minutes, and seconds. Faced with the possibility of a DGE, this type of operation does not make sense, but what is needed is the construction of new ways of exploring the relationships between angles and straight lines, for example (Bairral, 2019). The DGE also makes it possible to give greater visibility to the study of non-convex polygons and their relationships, something that is little appreciated in textbooks.

Because we believe in the potential of this geometry for construction and conceptual development, we will make a small explanation of some singularities of a DGE and the particularities of DGE in MTSD. Many authors have highlighted the characteristics and potential of a DGE[11]. For example,

[10] In a more current language, it is possible to enunciate the fifth postulate as follows: in the same plane, through a point exterior to a line, only one line parallel to the given line can pass.

[11] The characteristics that we present are universal, contemplated by a good part of the DGE, both in the desktop and MTSD versions (tablet or smartphone). However, our analysis is focused on the GeoGebra application in the geometry version for smartphones.

Gravina (2001) argues that such environments offer digital resources in which it is possible to build geometric objects from the defining properties. In addition, Laborde (1998) highlights the two-way action that experimentation with a DGE provides. By varying the elements of a geometric object, the subject obtains the feedback of the application, as this resource allows for feedback for the elaboration of conjectures, something that does not happen in a dynamic with pencil and paper.

Also, Arzarello et al. (2002) highlight the possibility of varied experiments in the construction of a DGE, which, in addition to allowing a broader view of the same object, can contribute to the process of proving a conjecture by giving the user the possibility of explaining it, to identify properties. The authors also argue that students appropriate the different handling modalities (entrainment) with different purposes, such as exploring, formulating, and validating conjectures.

Through GeoGebra, one can explore the validity of relationships and theorems. Furthermore, in verification in an DGE it is possible to raise a new curiosity in the learners, questioning them about the validity of a result and challenging them to other functions performed by the demonstration: explanation, discovery, verification, intellectual challenge, and systematisation (DeVilliers, 2001).

Laborde and Capponi (1994) emphasise that the mathematical knowledge presented in a DGE, due to the limitations of the software or the device, has a peculiar functioning and, in some cases, is different from that knowledge used as reference. In relation to this fact, we will exemplify. It is recurrent, in analyses involving interior collateral angles, that students have difficulties assuming the mathematical relationship involved (supplementary angles), in the case of an arithmetic analysis, given a particularity and limitation of GeoGebra regarding the number of decimal places, which makes it frequently impossible for the sum of the angles to be two right angles.

In addition to these characteristics, in a DGE, it is possible to transit between the particular and the general, as it allows the construction of a class of figures (Arzarello et al., 2002). It is also worth mentioning that in a DGE, since the constructions are not static, there are new alternatives before a geometric object, such as changing or dragging, which can enrich the creation and manipulation of a mental image, consequently contributing to the development of visualisation (Henrique, 2017).

We highlighted just a few characteristics of the DGE in computerised environments, and although they are universal, is touching a screen different from clicking with a mouse? According to Bairral (2020), touch-screen manipulations establish and inaugurate a field of manifestation of language and cognition. Finger and hand movements compose and shape our thinking. The author points out that, although some touches resemble actions performed to click or drag movements performed in a DGE on a desktop, they have differences in terms of orientation. While through the click, the action is mediated by a tool, in the MTSD, the action takes place from a continuous act, approaching object-subject.

Screen manipulations constitute different types of touches, such as single tap, double tap, press, slide, move, zoom in, zoom out, etc. Touching a screen and clicking on a mouse are different actions, as each one refers to sensory perception. On-screen manipulation constitutes a form of language, with particularities and implications for thought, as, like gestures, it represents forms of materialisation of thought in the communicative act. On the other hand, if some gestures can occur spontaneously, without apparent intentionality, the on-screen manipulations are specific, situated and intentional movements, the scholar adds (Bairral, 2017).

While manipulating an object on the screen of a MTSD, we perform a set of movements, some related to specific mathematical concepts, such as, for example, enlarging or reducing a figure. Dragging with a click one of the vertices of the figure or touching to "stretch" the diagonal with two fingers are epistemologically distinct actions, although both involve the diagonal method (Bairral et al., 2017).

Arzarello et al. (2014) identified two manipulation domains in the cognitive processes that are articulated during the action: the constructive and the relational. While the constructive scope involves basic actions, generally linked to constructions and the reorganisation of the geometric object, the relational scope has a more conjectural nature, with the intention of analysis, and involves other external elements, such as interactions and argumentation. In the constant movement of geometric reasoning in these two fields, processes such as visualisation, representation, and conceptualisation develop.

The specificities of a DGE that we present represent contributions to geometry teaching and learning, and understanding that the act of touching the screen is different from clicking with a mouse implies looking at the DGE in MTSD from a new perspective, in addition to enabling the analysis of how the conceptual development process takes place from the GeoGebra application.

Our hypothesis is that, when carrying out an activity, the mediation based on the analysis of a geometric object in a DGE through manipulations on the screen evidences the emergence of properties, intensifying the development of visualisation, as the subjects construct, modify, (re)create, and share touches and ideas in a joint action, all of this as an extension of thought in the act of touching the screen. In this way, they dialogue, argue, and share ideas, enhancing conceptual construction and development.

So far, we have discussed, among other topics, the role of visualisation, an essential cognitive skill for everyday life and intrinsic to the processes of mathematical reasoning and conceptual construction and development. We highlighted the importance of studying the relationships between angles and straight lines and pointed out that the DGE provide a new geometry. We also highlight how the DGEs in MTSD differ from those in computerised environments and how visualisation and conceptualisation can go hand in hand. In the following section, we present a teaching episode in which two pairs of students interacted, manipulated in/with MTSD, and visualised and intuited relationships between angles and lines on the smartphone through the GeoGebra application[12].

TEACHING EPISODES

The didactic situation we selected is part of a broader study[13] (Henrique, 2021), in which the methodological approach adopted was design experiments, a research methodology capable of dealing with the range of complex elements in teaching practices. This approach involves a particular form of learning and a systematic study, which can enhance learning within particular contexts in which various resources (artefacts) are used as support. (Cobb et al., 2003).

The objective was centred on the analysis and identification, through the GeoGebra application in smartphones, of the relationships between straight

[12] Geometry Version – 5.0.485.0. Available in: https://www.geogebra.org/download. *Link* tested on Feb 28. 2022.

[13] The investigation is part of the research project "Construindo e analisando práticas educativas em educação matemática com dispositivos *touchscreen*" [Building and analysing educational practices in mathematics education with touchscreen devices", funded by CNPq, and approved by the Research Ethics Committee of UFRRJ under opinion number 604/2015.

lines and angles – particularly in parallel straight lines cut by a transversal – by students of the 8th grade of elementary school of a state public school in the city of Resende, Rio de Janeiro.

The study was carried out during the first semester of 2018 and used the following procedures for data collection: (a) audio and video recording, (b) screenshot of the touch-screen manipulations of the smartphones used by students, (c) written responses from worksheets, and (d) researcher notes. In summary, the proposed task presents the following statements (Figure 3).

Figure 3

Summary of the tasks. (Henrique, 2021)

Protocol 2 task
2.1. Measure the angles on the same side as the transversal line. Do this procedure for all possible combinations (the angles outside the parallels, those between the parallels, etc.). Are there relationships between the angles? If so, which ones?
2.2. Measure an angle on each side regarding the transversal line. Is it possible to establish any relationship between the angles? Explain.
2.3. Investigate other relationships between pairs of angles that can be formed. As always, record your observations and, if necessary, make a drawing to clarify your findings.

To analyse this activity, we selected events from a video[14] of 17 minutes and 46 seconds obtained from the screen recording[15] (Bairral et al., 2022) of the smartphone used by students GD (12 years old) and MV (13 years old). The analysis looked for elements in the subjects' touches and speeches (writings, audios, screen constructions, etc.) that allowed us to investigate how the

[14] Refers to (conventional) video recording with audio from the capture of external manipulations, with a camera focusing on the movements and gestures of the subjects' hands when touching the screen.

[15] Capture direct touches on the device's screen through *MyAppSharer* app.

processes of constructions and movements on the screen occur and the development of the concepts addressed[16].

RESULTS: FROM ISOLATED CONCEPTS TO ARTICULATED LOOKS AND MOVEMENTS

Figure 4 shows the transcription of the audio and the objects of analysis of an excerpt from the video in which the students discuss the relationship between angles that are on the same side of the transversal (corresponding and collateral) and the properties involved (congruent or supplementary). Let us see the records obtained from this action.

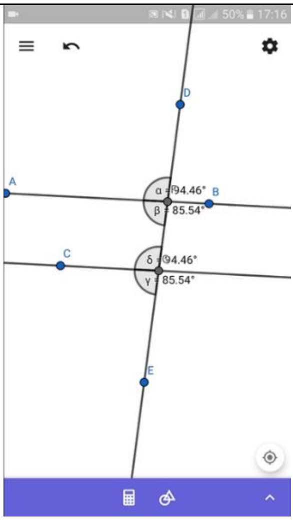
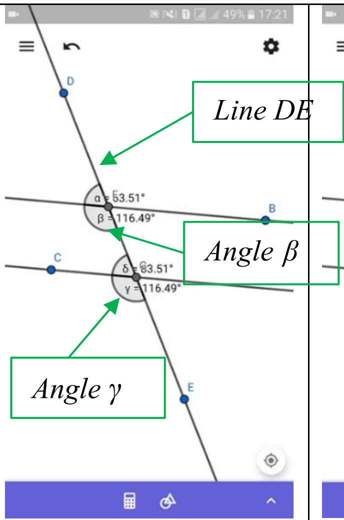
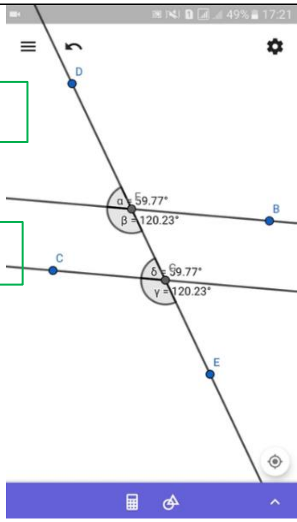
Through the screen recording, we observed that the learners established the relationship between corresponding angles based on congruence, a fact contemplated by negotiations and touches (Instant 00:16:25, for example, angle β = 116.49° and angle γ = 116.49°). We identified that the touches are articulated in constructive and relational domains (Arzarello et al., 2014), as the students formulated a conjecture based on handling the object in the DGE. It was verified and refined by them (*Look... here is one hundred and sixteen point forty-nine, and here is one hundred and sixteen point forty-nine, it doesn't change, look*). The fragment in Figure 4.1 summarises this reflective moment of the duo.

Figure 4

Actions of GD and MV students in the study of corresponding angles

Description
GD and MV manipulating and talking to each other about the position of the angles.

[16] See Henrique and Bairral (2020) for an analysis highlighting the importance of metaphors.

Printscreen		
		
Time 00:14:53	Time: 00:16:25	Time: 00:16:53

Transcription

<u>0:14:53</u> – MV: What is the similarity between those, huh?

GD: This one?

MV: Yeah!

MV: They are both outside, man! That's it, I guess.

GD: They are on the same line.

MV: Yeah, man... that's a relationship between them, man. They are on the same line. (Silence for a moment).

GD: So they're on the same line and... (inaudible excerpt) on the transverse line. (Silence, then the duo moves the construction).

GD: Congruent is like this: when you move it, is it the same thing?

<u>00:16:25</u> – MV: It's the same thing. Like... (the student moves the line DE) look, it's connected to D and E (points D and E, which represent, according to the student, the angle β).

MV: Here... it's on this side, this left part. Look ... here is one hundred and sixteen point forty-nine and here is one hundred and sixteen point forty-nine, it doesn't change, look (showing the relationship with the corresponding angle γ. Next, the student modifies the construction to emphasise his observation).

MV: They're the same, they don't change: one hundred and eleven and one hundred and eleven.

GD: Oh, yes!

MV: It doesn't change.

GD: But this one is the transversal one, right?

MV: What?

GD: It's the transversal that's moving, right? (Highlighting the manipulation performed by MV).

GD: So, we can put it like this, look: ... (referring to what should be written in the task).

00:16:53 – MV: They are congruent angles, that's all (referring to the correspondents).

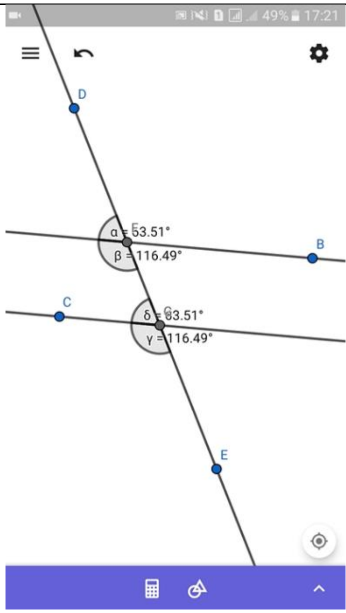
In addition to the articulated conceptual understanding, the transcription of Figure 4.1 also highlights the idea of reference emerging in the students' reasoning, that is, which elements they are considering in their analysis, moment-to-moment time and angles.

In this new endeavour, the students analysed interior collateral angles, as shown in the following records. In the excerpts, the students discuss the relationship between angles on the same side of the transversal line (corresponding and collateral) and the properties involved (congruent or supplementary). In Figure 5, we present the records obtained from this action.

Examining the excerpt allowed us to verify that the pair used the drag touch with a double function: constructive and relational. The first function aimed at facilitating the analysis; for this they interchanged lines CG and AB (Time: 00:16:53 → Time: 00:17:15), which meant that only the lines and the interior collateral angles (Time 00:17:15, angles α and γ), and displaced the line CG towards the line AB, bringing the angles α and γ closer together. The movement of dragging the action generated a new construction in which only the analysis objects were. Thus, in the second function assigned to the dragging movement, the students formulated a conjecture about the relationship between angles α and γ (supplementary angles, Time 00:17:29).

Figure 4.1

Summary from Figure 4.

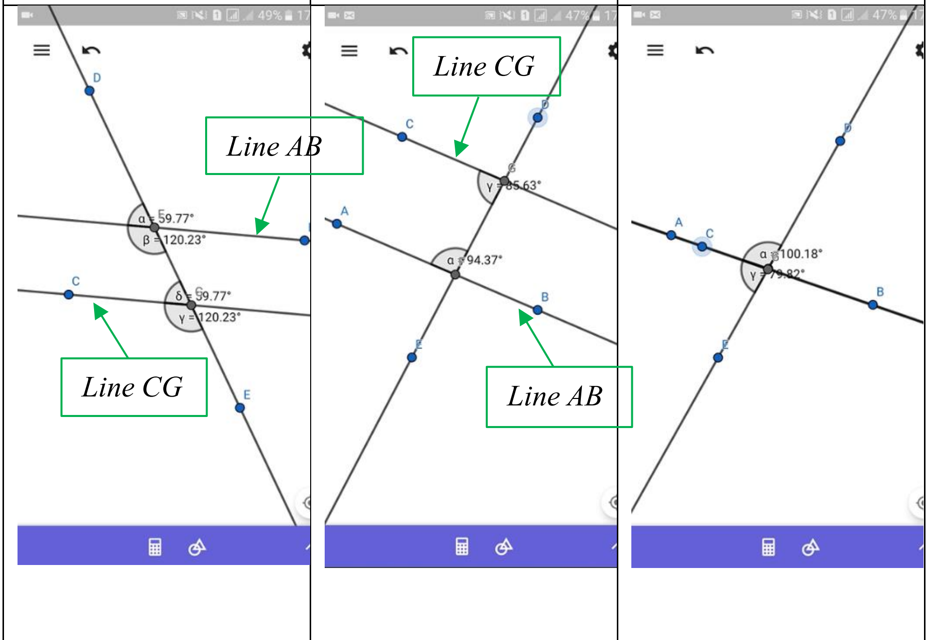
Transcription	**Description**	***Printscreen***
GD: Congruent is like this: when *you move it, is it the same thing*? 00:16:25 – MV: It's the same thing. Like... (the student moves line DE) look, it's connected to D and E (points D and E, which represent, according to the student, angle β). MV: Here...*It's on this side, this left part*. Look ... here is one hundred and sixteen point forty-nine and here is one hundred and sixteen point forty-nine, it doesn't change, look (showing the relationship with the corresponding angle γ. Next, the student modifies the construction to	congruence of angles the movement of line DE and the alteration of external collateral angles the correspondence and congruence of angles, and the left side as a reference	

emphasise his observation).		

Figure 5

Students' actions in the analysis of interior collateral angles.

Printscreen		
Time: 00:16:25	Time: 00:17:15	Time: 00:17:29



Description
GD and MV manipulating and visualising the relationships between angles

The students could have established a conjecture through an arithmetic analysis, adding the α and γ angles in the "Insert" field of the GeoGebra application. We do not know whether this process would lead students to the conceptual elaboration because, as we have already highlighted, in this type of operation, given a particularity of the application – a restricted number of decimal places – the sum approaches 180º. However, we observed that the students appropriated a particularity of a MTSD, in which the act of dragging involves different purposes, such as formulating and validating conjectures, to establish a relationship between the angles α and γ, translating the line CG, overlapping -a over AB, making them supplementary, that is, they instituted a visual conjecture to fit their purpose, which is a type of reasoning in the relational scope.

The excerpt from the episode allowed us to identify the strategies developed by the students and the manipulations articulating between the constitutive and relational domains (Arzarello et al., 2014). The students' action of translating one of the lines in the object of analysis to confirm that the sum of angles α and γ results in two right angles suggests insight as an element of mathematical reasoning and reveals that visualisation was enhanced by touching the screen of the smartphone, intensifying the development of the concept of collateral interior angles.

From this event, we drew conclusions about the visualisation enhanced by the correlated touches in the construction and conceptual development. In this case, the manipulation of the mental image involved the representation of the concept (Zimmermann & Cunningham, 1991). It is worth complementing that the composition of the touches includes the visualisation and graphic representations (constructions on the screen and their movements), elements that characterise ways of reasoning in MTSD.

CONCLUSION

We discussed the importance of visualisation, representation, and conceptualisation – processes intertwined in learning with DGE. We highlighted the importance of studying the relationships between angles and lines, the role of another geometry provided by the DGE, and some characteristics and singularities of those environments.

MTSD represent a physical extension of our bodies (Bairral, 2020), and to a certain extent, touching the screen influences the conceptual images we create, as there are singularities in sensory and motor terms in which the referred action influences visual perception. Visualisation is enhanced by touches on the screen, as well as the construction and conceptual development linked to this process. At this core, conceptualisation and geometric visualisation can be related to the emergence and identification of mathematical relationships through activities in an DGE with MTSD, since the conceptual elaboration and visualisation of constructions on the screen demand tasks that allow for continued experience and different ways with the object(s) under analysis.

Approaching mathematical concepts through touches can break the hierarchy established in Euclidean geometry (axiom/definition/properties) and highlights the relevance of geometric learning mediated by a DGE in MTSD from the emergence of concepts and properties potentiated by the visualisation of non-static shapes. With this, we defend that those devices can create a more fluid and articulated curricular organisation of geometric concepts and that the activities also allow the emergence of new concepts.

As exemplified in the data from GD's and MV's manipulations, the work with lines and angles in touch-screen manipulations for formulating conjectures allowed identifying the emergence of a new concept in the study of relationships and properties between parallel lines, transversal lines, and formed angles: the concept of reference. The emergence of this concept is the result of the dynamics of the simultaneous movement of objects on the screen. With the multiplicity of synchronous movements, the examination of the construction of a conjecture (and its validation or refutation) happens when having as reference the mathematical entity to be considered, which can be a straight line or an angle. That is, what lines and angles are considered? Therefore, further studies focusing on the analysis of the concept of reference are in order.

AUTHORS' CONTRIBUTION STATEMENT

MPH and MAB conceived the idea. MPH conducted the field research, collected the data, and performed the analyses under MAB's supervision. Both actively discussed the results and revised and improved the final version of the work.

DATA AVAILABILITY DECLARATION

Data supporting the results of this study will be made available by the corresponding author, MPH, upon reasonable request.

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