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Exploring Students Reversible Reasoning when Sketching an Original Graph Involving Derivatives Concept

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ABSTRACT

Background: Reversible reasoning, a key aspect of mental activity, is important for students at all levels of mathematics. **Objective:** The purpose of this study was to characterise students' reversible reasoning when sketching graphs involving derivatives. **Setting and participants:** The data for this research was generated from a qualitative approach. We conducted clinical interviews with four students aged 18-19 who had completed and graduated in advanced calculus courses. They were selected for their high scores in the course. **Data collection and analysis:** Think aloud methods and task-based interviews were used to collect data. The analysis covered two original graph sketch assignments involving the graph of the derivative and its properties. **Results:** Through data analysis, we discovered two characteristics of reversible reasoning: Initial Reversible and Ongoing Reversible, which provide the initial framework for future research. These two characteristics are rooted in the student's perspective of the problem at hand, where the Initial Reversible student is dominated by the accommodation process, while the Ongoing Reversible student tends to experience cognitive conflicts that cause him to change his direction of thinking. **Conclusions:** We discuss the implications of our findings for future teaching and curriculum development in calculus.

Keywords: Reversible reasoning; Graphs original; Graphs derivative; Analytical derivative; Calculus

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Explorando o raciocínio reversível dos alunos ao esboçar um gráfico original envolvendo o conceito de derivadas

RESUMO

Contexto: O raciocínio reversível, um aspecto fundamental da atividade mental, é importante para os alunos em todos os níveis da matemática. **Objetivos:** O objetivo deste estudo foi caracterizar o raciocínio reversível dos alunos ao traçar gráficos envolvendo derivadas. **Cenário e participantes:** Os dados para esta pesquisa foram gerados a partir de uma abordagem qualitativa. Realizamos entrevistas clínicas com quatro alunos, com idades entre 18 e 19 anos, que haviam concluído e graduado em cursos de cálculo avançado na época da entrevista. Eles foram selecionados por terem pontuações altas no curso. **Coleta e análise de dados:** Métodos de pensar em voz alta e entrevistas baseadas em tarefas foram usados para coletar dados. A análise cobriu duas tarefas de desenho de gráfico original envolvendo o gráfico da derivada e suas propriedades. **Resultados:** Por meio da análise de dados, descobrimos duas características do raciocínio reversível: Reversível Inicial e Reversível Contínuo, que fornecem a estrutura inicial para pesquisas futuras. Essas duas características estão enraizadas na perspectiva do aluno sobre o problema em questão, onde o aluno Reversível Inicial é dominado pelo processo de acomodação, enquanto o aluno Reversível Contínuo tende a vivenciar conflitos cognitivos que o levam a mudar sua direção de pensamento. **Conclusões:** Discutimos as implicações de nossas descobertas para o ensino futuro e desenvolvimento de currículo em cálculo.

Palavras-chave: Raciocínio reversível; Gráficos originais; Derivada de gráficos; Derivada analítica; Cálculo

INTRODUCTION

Reversible reasoning is one of the mental activities required to solve mathematical problems. Such as when students reconstruct the problem from input to outcome (Tunç-Pekkan, 2015) looking for the missing value of $4 + \cdots = 7$, looking for values for x that satisfy $14 - \left(\frac{15}{7-x}\right) = 9$, or looking for θ that satisfy $\sin\theta = \frac{1}{2}$. Other literature reveals that mental activity can stimulate students to build relationships between two concepts (Haciomeroglu et al., 2010; Paoletti et al., 2018), for example, addition and subtraction, exponents and logarithms, derivatives, and derivatives in the fundamental theorems of calculus. Next, through reversible reasoning, students are encouraged to reconstruct the direction of thinking from thinking directly to the opposite direction (R. Kang & Liu, 2018). Suppose students have understood that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, then to solve $\cos 30° \sin 15° + \cos 15° \sin 30°$ they had to change the direction of their thinking. Furthermore, one of the differences between reversible reasoning and other types of reasoning (e.g. covariational, quantitative, or algebraic reasoning) is that through reversible reasoning students are able to minimise the complexity of the problem by involving reversing situations, operations, relationships, or its representation. Students can perform in-depth analysis by recalling the information stored in their memory.

The absence of reversible reasoning results in several problems for students. For example, students find it easier to convert $3\frac{2}{3}$ into an improper fraction, than to represent $\frac{17}{5}$ in a mixed number (Norton & Wilkins, 2012). Ramful's findings (2014) indicate that the absence of reversible reasoning makes it easier for students to apply the $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ rule than vice versa. It is easier for students to draw a graph than identifying a visual representation of a graph, to the point of generalising the property of inverse function $f^{-1}(f(x)) = x$ for all cases. Therefore, reversible reasoning becomes a mental activity that students need to develop. It encourages students to involve anticipation, observing situations from both sides, and re-abstract the results of previous experiences to build new knowledge.

The importance of the study of reversible reasoning has motivated several researchers to carry out investigations with different findings (Hackenberg, 2010; Ramful, 2014; Simon et al., 2016; Steffe & Olive, 2010). However, most of these findings reveal that the numerical nature of the problem situation causes students to make reversible reasoning. Therefore, the existing literature is limited to operational aspects, and there are still few investigations on its conceptual aspects (e.g. functions and inverses, and calculus). This is because students not only construct the operation of a problem, but they also need to construct its inverse (Lamon, 1993). However, as the researcher found, there is only one study related to reversible reasoning for the conceptual aspects written by Hachiomeroglu et al. (2010). Therefore, one important aspect to note is how problems stimulate students to involve reversible reasoning.

In this article, we identify three facts: first, the literature on reversible reasoning is limited to operational aspects; second, there is little literature that discusses reversible reasoning with regard to problems in calculus; third, the Indonesian curriculum requires students to build reversible reasoning, but it has not been implemented in the classroom. These three facts led us to fill the gap by exploring the reversible reasoning students perform when they complete a calculus problem. We are interested in how students sketch a graph of a parent function when given a graph of its derivative and its properties. This is in line with the recommendations of previous research (Simon et al., 2016) that the topics in calculus are constructed by involving reversible reasoning, so that students can interpret tables, graphs, and diagrams.

Apart from the importance of topics in calculus, attempts to integrate many concepts have become a complex problem for students, especially in graph sketching (Fuentealba et al., 2017; García et al., 2011). Most students tend to involve algebraic techniques when sketching graphs (García-García & Dolores-Flores, 2019; Hong & Thomas, 2015). This, in fact, does not help them in sketching the graph of the function from the graph of its derivatives or its properties (Sánchez-Matamoros et al., 2014). In this problem, they need to develop ideas about the derivative of a function based on a few points, namely geometric interpretation, recognising conditions in which the function is differentiated at a point, or intervals that allow students to identify conditions where the function is increasing or decreasing, maxima/minima, inflexion points, and the concavity of a graph. However, drawing a graph of a parent function given a graph of its derivatives or its properties is not trivial. This causes students to have difficulty connecting the slope of the tangent to its derivative without involving algebraic expressions (Natsheh & Karsenty, 2014) and connecting the meanings of the second derivative when drawing graphs (Sofronas et al., 2011). This is because students' thinking preferences are dominated by analytic thinking, making it difficult to apply derivative properties in describing graphs (Haciomeroglu et al., 2013). In addition, the situation is related to reversible reasoning, which will allow students to connect the two concepts and contribute to developing their ability to complete graphic assignments.

Another fact is that only some of the students managed to involve reversible reasoning in sketching graphs. In this case, we wondered how to characterise the reversible reasoning that students generate when working on graphic assignments. Therefore, this study has three contributions to the field of mathematics education: first, it provides a framework for studying students' reversible reasoning in sketching graphs of functions, where the characteristics produced by students have not been identified. Second, research on students' reversible reasoning can be used to design classroom learning. Third, the results of this investigation have direct implications for the development of the calculus curriculum, particularly regarding graphic sketching.

Therefore, in this article, we propose to answer the following research questions:

"*What are the reversible reasoning characterisations when students sketch graphs of functions involving derivatives?*"

METHODOLOGY

The data used in this research comes from studies with a qualitative approach. Therefore, we used task-based interviews and think aloud as data collection methods. In a task-based interview, one subject meets an interviewer who introduces the task to the subject in a pre-planned way. The interviewer uses an audio or video recorder to capture the verbal expression and sometimes records the subject's mental activity while solving problems, which will be analysed later (Mejía-Ramos & Weber, 2020). Therefore, task-based interviews are semi-structured in the sense that the interviewer has a number of pre-planned questions to ask the subject during the interview. Furthermore, in this section, we will describe the following: the research context, the participants, the methods used to collect data, the design of the instruments used, and the methods used to analyse the data.

Context and Participants

This research was conducted at a university in the eastern part of Java, Indonesia. It is a public university with 1000 students (estimated) enrolled across four levels of education. The curriculum used at this university follows the Indonesian National Qualifications Framework that encourages students to develop their knowledge, problem-solving skills, and research environment. At universities in Indonesia, students usually study calculus in three different semesters. In the first semester, students learn basic calculus, where the material studied is pre-calculus, limits, and derivatives. In the second semester, they study integrals, where they learn the concepts of antiderivatives, indeterminate integrals, and the fundamental theorems of calculus. In the third semester, they study integration techniques and advanced calculus. Students need to underline that when they study integrals, they need to think about the relationship between derivatives and antiderivatives.

As part of our longitudinal research, we conducted clinical interviews for one hour separately with four students, namely Albertin, Taufiq, Dina, and Nizi (pseudonyms). They are 18-19 years old. All four students were enrolled as fourth-semester students and had recently completed an advanced calculus course with flying numbers. So, we assume that they are familiar with the tasks developed in this study. This was proven when they completed the tasks designed in this study, where the four were able to involve reversible reasoning by establishing a relationship between derivatives and antiderivatives in sketching a graph of the function. Meanwhile, for students who cannot sketch the graph, we provide them with a few limited interventions to find solutions. However, most of them are unable to find a solution because their knowledge is limited to graphic sketches with known functional formulas. These results can be traced from our previous study in the article of Ikram et al. (2020). For students who are unable to complete the assignment, we make it a consideration for further research

Instruments to Collect Data

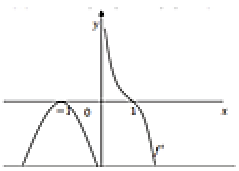
We used two complementary methods to collect data, think aloud and task-based interviews. Think-aloud method allows students to express their mental activities, while task-based interviews are carried out to clarify things that are not visible during problem-solving using the think-aloud method. We also used interviews to clarify the unique findings that the subjects showed while solving the problem. First, we asked the four students to think aloud during problem-solving. We chose this method because think-aloud refers to a subject's verbal expression that is focused on a particular activity without any intervention that interferes with mental activity and helps express the subject's thoughts (Leiss et al., 2019). Because this method is not familiar to students in Indonesia, we trained them first before they completed their assignments so that they could get their thoughts out. In addition, we did not interfere with their thought processes and gave them the freedom to provide information when solving problems. Second, we used task-based interviews, aiming to reveal things that were not visible when students were thinking aloud. This method helps observe, register, and interpret the subject's behaviour, including the verbal expressions expressed by the subject (Goldin, 2000). Furthermore, the interview protocol was semi-structured and anticipated unexpected possibilities when the subject completed a task.

All subjects completed assignments individually in a closed room with only one student and researcher present. When they completed the assignment, the researcher asked several questions that had not been seen when they were thinking aloud in order to refine how and why each student completed the task in this way. Typical questions in our interview guide weree: "What do you think?", "Can you provide other reasons?", "Can you show it on the graph?" and "Why are you silent/restless after completing it? ". We also asked additional questions in the form of: “What is the meaning of any information provided on the assignment?” (e.g., graph interpretation of f'); “What knowledge do you need to sketch the graph?”; and “How can you use that knowledge to solve problems”.

The tasks that the subjects were required to carry out are presented in Table 1. As we can see, the assignment does not allow the subjects to look up the formula for its function. In the following, we briefly describe the purpose of each proposed task.

Table 1

Task Descriptions

Tasks	Characteristic
Task #1. Given that f' is the derivative of f. The graph of f' is presented in the following figure. Sketch the appropriate f chart! 	Students should think about situations that result in: (1) f' positive for the interval (0.1); (2) f' is negative for the intervals (-∞, 0) and (1, ∞); (3) f' increases for the interval (-∞, -1); (4) f' decreases for the intervals (-1,0) and (0, ∞); (5) f' intersects the X-axis at $x = -1$ and $x = 1$ or $f'(-1) = f'(1) = 0$; and (6) f' is not continuous at $x = 0$.
Task #2. Sketch a graph of the function f satisfying the following properties: ✓ $f'(1) = f'(5) = 0$; ✓ $f'(x) > 0$ *for* $-2 < x < 1$ *and* $5 < x < 7$; ✓ $f''(x) > 0$ *for* $3 < x < 7$ *and* $x > 7$; ✓ $\lim_{x \to -\infty} f(x) = -\infty$ *and* $\lim_{x \to -2} f(x) = -\infty$	Students should think of a graph of f satisfying the four properties given.

In **task 1**, students were given a graph of the derivative that included the increasing/decreasing of the f' curve, the f' curve intersects the X-axis at $x = -1$ and $x = 1$, the extreme values, and the discontinuity of f' at x = 0. Students should sketch an initial graph, and the goal was to assess whether they could see the problem situation from the reverse direction by coordinating the f' sign with monotony f, the behaviour of f' with the concavity of f, change in behaviour of f', with extreme points and the intercepts of f, and the discontinuity of f' with the points where f is non-differentiable.

Task 2 provides analytical information for sketching a graph of f. Here, students were given some information that stimulated them to think about the possibilities of the resulting graph. The purpose of this assignment was to ask students to sketch a graph of f by identifying the first derivative's sign in an interval, the second derivative's sign in an interval, and the meaning of the limit value of the function. On the other hand, another objective is to observe whether students can identify contradictions in the analytical situation generated. This task stimulated the subjects to think of a function f that could satusfy the four given properties and information that was not given in the task.

Data Analysis

We emphasise that the process of data collection and data analysis in this study is based on a constructivist point of view. The viewpoint of the constructivist theory states that students' knowledge consists of a set of schemas based on previous experience (Dubinsky, 2002; Von Glasersfeld, 1995). This view implies that we do not have direct access to students' knowledge and can only model their interpretations based on think aloud and observed interview results (e.g. verbal expressions, behaviour/gestures, and generated graphics). Thus, our analysis reflects our best efforts at characterising students' reversible reasoning in sketching graphs involving derivatives. Our data analysis is in line with Corbin and Strauss's (2010) description of grounded theory, in which the reversible characterisation of student reasoning emerges from data analysis. This analysis consists of three stages as follows.

Preliminary Analysis. Initial data analysis began after a think-aloud and interviews were conducted. In this case, we made initial guesses based on the students' verbal expressions, gestures, and graphical sketches. This initial guess was used to guide follow-up questions from the interviewer. Based on students' responses to follow-up questions, initial guesses were corrected or changed. Additional questions were asked until enough data had been collected. After conducting one interview with students, the research team met to explain the results obtained. The more discussions that were carried out, the researcher could better pattern the characteristics of students' reversible reasoning. In particular, after conducting several interviews, the research team found that two tendencies of reversible reasoning occurred when students sketched the charts. So, we paid more attention to situations that triggered students to make reversible reasoning and clarified students' verbal expressions when thinking aloud through interviews

Open Coding. After conducting the interviews, we analysed the results of the student-to-student transcripts and video recordings to characterise students' reversible reasoning from the two given tasks. Therefore, we made open coding of students' interpretations that emerged because of their mental actions in any problem-solving process. We analysed the results of the transcripts by developing codes to describe relevant parts of students' reversible reasoning while sketching the graph. Furthermore, we refined the codes by adding and expanding the initial coding to characterise students' reversible reasoning. This process continued until we analysed all student data. At the end of this open coding process, two codes stood out: initial reversible and ongoing reversible. They emerged as a broad characterisation of students' reversible reasoning when sketching graphs involving derivatives.

Axial Coding. Once these initial reversible and ongoing reversible categories were defined, we refined these two findings using axial coding. To refine the definitions of the two categories, we compared provoking situations and verbal expressions that indicated the involvement of reversible reasoning. Next, we compared students in different categories to develop more detailed descriptions of our analysis. Finally, we re-coded the transcript using the refined code. Finally, we used these two categories to frame findings of the reversible characterisation of students' reasoning in sketching graphs involving derivatives.

From the analysis, we performed a reliability test, in which each research team was also involved in analysing the results of student transcripts separately. Then each of these analyses was aggregated and compared to discuss any differences that emerged. After an agreement was reached, a revision was made to the resulting categories. In addition, the trustworthiness was enhanced by ensuring that the data collected was accurate and complete by making word-for-word transcripts of each interview and validating the coding process down to the categories found with several mathematics education experts.

RESULTS

This study aimed to characterise students' reversible reasoning when sketching graphs involving derivatives. From our findings, we observe that some students (Albertin and Taufiq) are immediately aware of the relationship between derivatives and antiderivatives, for example, the relationship between the value of $f'$and the increasing/decreasing behavior of f at an interval, the relationship between the indifferentiability of f with the continuity/discontinuity at x = a, the relationship between the sign of the first derivative and the monotony of f over an interval, the relationship between the sign of the second derivative and the concavity of f in an interval, and the meaning of the limit notation with behaviour f.

In addition, other students (Dina and Nizi) tried to find a function that gives the graph f' and the function f that satisfies in $f'(1) = f'(5) = 0$, $f'(x) > 0$ for $-2 < x < 1$ and $5 < x < 7$, and $f''(x) > 0$ for $3 < x < 7$ and $x > 7$. The two students collided with a complex situation that resulted in a change in the direction of thinking so that they realised derivatives properties to solve the problem.

In short, we observe students' awareness to relate the properties of derivatives by reversing the problem situation. The properties that they use are: f' is positive, f' is negative, f' is 0, f' does not exist, and uses conditional relationships (e.g., the relationship between the sign of the first derivative and the behaviour of f, the relationship between the signs of the second derivative and the concavity of f, the relationship between the change in f' and the extreme or inflexion point, the relationship between the indifferentiability of f with the continuity of f) to obtain a graph sketch of f. These steps refer to the initial reversible characteristics. Meanwhile, the ongoing reversible characteristics show that students' thinking is influenced by previous experiences. Furthermore, the inability to continue the completion process and their awareness that the initial idea was not enough to sketch the actual graph resulted in a change in the direction of students' thinking.

Initial Reversible Characteristics

Part I: Students' interpretation is dominated by visual thinking

The two students showed different responses when sketching an f graph involving a derivative graph. Albertin was silent for a long time (about 2 minutes) while observing the f' graph. By underlining some information on the task (e.g., "f' derivative of f", "graph of f'", and "sketch the graph of f), he showed an indication of reversing the problem situation by looking for graph f resulting in graph f'. Furthermore, Albertin recognised some of the meaning of the relationship between the value of f' and the behaviour of f, namely, when f' is positive, then f increases. Further, it begins to coordinate between the value f', the location of f', and the behaviour of f, namely: when f' is negative, or the curve of f' is below the X-axis, then f is a descending function; when f' is 0 or f'intersects the X-axis, then f has an extreme value or f is stationary; when f' is positive, or the curves of f' is above the X-axis, then f is an ascending function; and f' is not continuous at x = 0, because as x approaches 0 from the left, the value of $f'$is approaching -∞ and as x approaches 0 from the right, the value of $f'$is approaching + ∞.

Unlike Albertin, Taufiq immediately realised the meaning of the graph f' representing the gradient of f. Next, he divided the X-axis into three intervals, namely, $(-\infty, 0)$, $(0,1)$, and $(1, +\infty)$. He then integrates the relationship between the location and the value of f', namely: curve f' for the intervals (-∞, 0) and $(1, +\infty)$ is negative because curve f' is under the X-axis; curve$f'$for interval (0.1) is positive because curve f' is above the X-axis; the value of f' at point x = -1 and x = 1 is 0 because the curve f'intersects the X-axis at that point; and f' has no value at the point x = 0 because the curve f' does not intersect the Y-axis and is not continuous. Taufiq further interpreted the behaviour of the curve f'around the point x = 0, i.e., as x gets closer to 0 from the right, the function value gets bigger, and as x approaches 0 from the left, the function value gets smaller. The following transcript contains a verbal explanation of the two students' interpretations of f graph sketches involving derivative charts (See Table 2**)**.

Table 2

The transcript contains Albertin and Taufiq's verbal explanations

Think Aloud	***Description***
Albertin: *oh ⋯from here it appears that if the graph f' is the derivative of f, usually it represents when f' is positive then f will increase. From the graphic, for example, if the value is negative or below the X-axis, it means that the graph is going down. if, for example, it is zero, then it is at the peak point, the vertex is a stationary point. Then, if it is above the x-axis, it means that the function will increase. If in the discontinuous part it means the derivative approaches infinity and minus infinity*	• Reversing the problem situation: Finding the f graph resulting in the f' graph • Recognizes the relationship between the value of f' and the behavior of f in an interval • Interpret f' behavior around $x = 0$
Taufiq: *From this graph, if the derivative graph is known, then the graph of its function is asked. oh⋯, it means that this graph is originally from graph f, so I immediately know that if the derivative is negative it means f is going down, if it is positive it means it is going up. Then, the graph of f'. 'this is a slope graph. For x = 0, this is not continuous because f' gets smaller from the right and gets bigger from the left as it approaches x = 0*	• Reversing the problem situation: initially there is a graph of f • Recognizing the relationship between the sign of the first derivative and the increase/ decrease of f • Interpret the rate of change in behavior of f'around x = 0

For the task of sketching f based on analytic properties, Albertin has caught the core of the problem; namely, the function f to be sketched must meet the four known characteristics. While Taufiq experiences a disequilibrium, where he only recognises some of the known properties (e.g., $f'(1) = f'(5) = 0$ indicates that f is stationary at x = 1 and x = 5 or the tangent f has a gradient of 0 at x = 1 and x = 5, and $f'(x) > 0$ in intervals of -2 <x <1 and 5 <x <7 indicates that f is increasing or the tangent f has a positive gradient for that interval) and some others that have not been able to be interpreted directly. Next, he tries to understand the effect of the sign of the second derivative and the limit notation of its function to obtain a graphical sketch of f. He tried to identify the consequence of $f''(x) > 0$, $\lim_{x \to -\infty} f(x) = -\infty$, and $\lim_{x \to -2} f(x) = -\infty$. The following transcript contains a verbal explanation of the two students' interpretations of the f graph sketch that involves its properties (See Table 3).

Table 3

The transcript contains Albertin and Taufiq's verbal explanation

Think Aloud	*Description*
Albertin: *Here we will sketch the graph that satisfies these properties, $f'(1) = f'(5) = 0$, $f'(x) > 0$ for the domains between -2 to 1 and between 5 to 7, so $f''(x)$ is more than 0 from x=3 to x= 7 and for x greater than 7. That means there is a graph that meets all four properties.*	• Reversing the problem situation: Finds a graph of f that satisfies all four known properties
Taufiq: *I imagined the first one like before, stationary as well as fluctuating. Since $f'(1) = f'(5) = 0$, it means that the gradient is stationary, and the gradient is 0. If $f'(x) > 0$, it means that the gradient is positive, so it increases for intervals of -2 to 1, similar to in the interval 5 to 7. If the other intervals, I don't think the same thing happens*	• Recognizing the relationship between the sign of the first derivative and the monotony of the graph f'

The verbal expressions on the transcript gave evidence of reversible reasoning that emerged when the two students interpreted the problem by reversing the situation, to the point they realised that the core of the problem involved the relationship between derivatives and antiderivatives.

Part II: The Anticipation Model that students present in graphic sketches

This section highlights ideas that students come up with after interpreting the problem. We can observe Albertin's analysis when working with tables (Figure 1). He compares the previously solved problems with the problems at hand and uses his knowledge of the definition of derivatives and continuity. Albertin sees the problem situation by identifying the situation at f, which causes f' to be discontinuous at x = 0. He states that the value of $f(x)$ is not the same as the value of f' at point x = 0; or the function value for the left limit and the right limit is different. This information is integrated until it is concluded that the discontinuity of f at x = 0 made f' is also not continuous at x = 0. This shows that Albertin's analysis of the behaviour f' at x = 0 is not comprehensive but has been used to generalise the problem.

Figure 1

Sketches made by Albertin

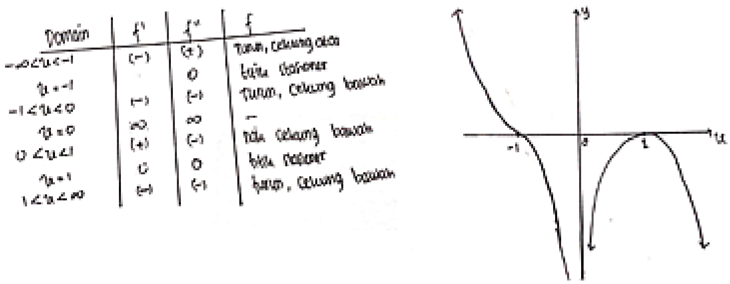
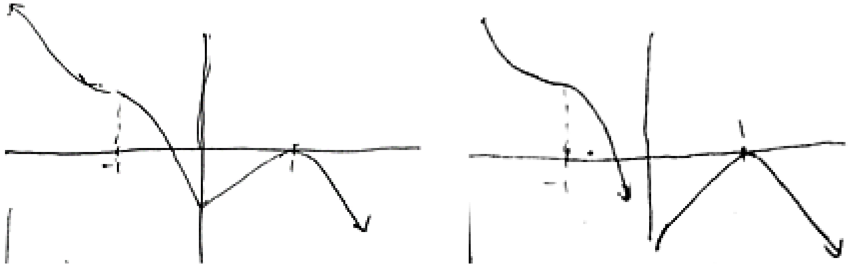


Figure 2

Sketches made by Taufiq



(a) (b)

Unlike Albertin, Taufiq works predominantly by visual thinking (Figure 2). In this case, we use more verbal expressions to sketch the graph of f, namely: since f' is negative in the intervals $(-\infty, 0)$ and $(1, \infty)$, then the tangent of f is negative, meaning f must be decreasing in that interval; since f' in the interval (0,1) is positive, then the tangent line is positive meaning f is increasing in that interval; since f' at point x = -1 and x = 1 is 0, then f must be stationary at that point, because the tangent is 0; Since f decreases from the left of x = -1 and f continues to decrease from the right side of x= -1, it means that f has a turning point at x= -1; since f increases from the left side of x=1 and decreases from the right side of x=1, it means that f is a local maxima at x=1; and since f' at point x = 0 does not exist or f is not differentiable at f, it means that two conditions may occur. It is either: curve f forms a fracture (sharp turn) at x = 0; or f is not continuous at x = 0, forming indefinite discontinuity, meaning the line x = 0 is the vertical asymptote of the graph f. Furthermore, Taufiq uses the relationship between changes in gradient values and changes in behaviour f. For example, f has a negative gradient in the interval $(-\infty, 0)$. As x goes to the right, f' approaches 0, until at x = -1, the gradient is equal to 0. On the right of x= -1, the gradient is decreasingly negative. The following transcript contains a verbal explanation of the students' anticipation model when sketching an f graph that involves a derivative graph (See Table 4).

Table 4

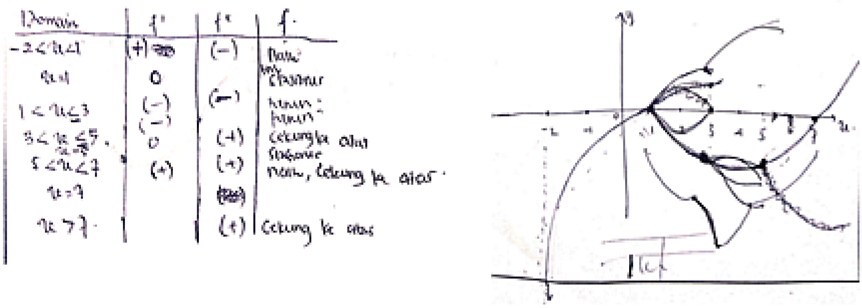
The transcript contains Albertin and Taufiq's verbal explanations

Think Aloud	*Description*
Albertin: *I think this is the case [Albertin points out x = 0] where it is moving towards infinite, what do we need to do so that it doesn't reach infinite? As far as I know that is the concept of limit. The limit has no value if the left limit and the right limit is different, it means that it has no value on this graph [Albertin points to graph f] and here [Albertin points to graph f ']*	▪ Finds out why f is not differentiated at x = 0
Taufiq: *Chart f decreases in the interval -∞ to -1, this is a negative gradient, as x moves towards x=-1, f' is getting closer to zero, at x = -1 f' is 0, then after that, it's negative again, so the graph goes down, continues like this [curve f flattens], from there, f' continues to decrease again. Then this one [graph f for the interval (0, ∞)], the gradient is positive, then to the right the positive is getting closer to zero, exactly at 1, f' is 0, after that the gradient is negative, so the graph goes up first and then continues to decrease*	▪ Using the relationship between changes in gradient values and changes in behavior f

To sketch a graph of f based on analytical properties, Albertin integrated the relationship between the sign of the first derivative and the behaviour of f and the sign of the second derivative with the concavity f. This is evident from Albartin's sketches in Figure 3. For the limit notation of the function, he takes a simple case (e.g., $\lim_{x \to 0} f(x) = -\infty$) to draw conclusions. As for intervals and situations that are not given, Albertin works trial and error and negates the problem situation (e.g., f' and f'' are not given in the interval $1 < x < 3$, so it can be concluded that $f'(x) < 0$ and $f''(x) < 0$).

Figure 3

Sketches made by Albertin



(a) (b)

Figure 4

Sketches made by Taufiq

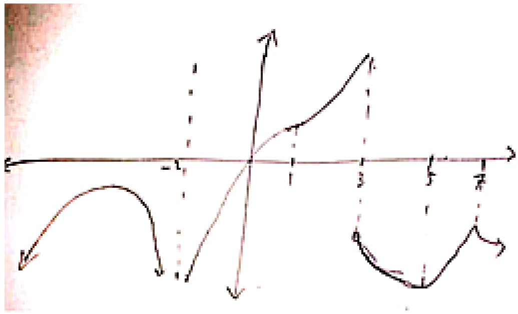


Table 5

The transcript contains Albertin and Taufiq's verbal explanations

Think Aloud	*Description*
Albertin: *1 to 3 is negative, it means it goes down, there is also nothing means negative too, if negative means that it is concave down*	• Negating the problem situation
Taufiq: *from x=-2 to x= 1, , f' is positive, the more to the right f' is getting smaller. now at x= 1, f' is 0. after that maybe it is positive or negative here, if f' is positive means that f increasing, correct? in x=1 it becomes a turning point, if it goes down it means, it becomes the local maximum, then from x=5 to x=7, at x=5, f' is 0, it continues to be positive, the more to the right, f' is getting bigger, meaning that here it must be negative*	• Predicts the behavior of the f curve for an interval which is not given.

The anticipation raised by Taufiq (Figure 4) for the second problem appears when analysing the second derivative sign and the limit function notation ($\lim_{x\to-2} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$). In this case, Taufiq changed the old scheme of using the second derivative sign to identify the local maxima/minima at a stationary point to the new scheme formed through the relationship between the sign of the second derivative of the concavity of f. While Taufiq's analysis of the meaning $\lim_{x\to-2} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ is carried out by integrating the behaviour of f with the limit notation of its function, namely: (1) f $\lim_{x\to-\infty} f(x) = -\infty$ represents that as x gets smaller, then the f curve is a decreasing function or when the f curve moves from left to right, f is an ascending function; and (2)) $\lim_{x\to-2} f(x) = -\infty$ represents as x approaches -2 from the left and right, so f is a decreasing function. From Taufic's mental activity, he also predicted the f curve's behaviour for intervals which is not given (see Table 5).

Verbal expressions on the transcript give evidence that the anticipation model that emerged between the two students is relatively the same, which is by building mathematical ideas from known properties, negating problems, predicting problem situations, and working on trial and error.

Characteristics of Ongoing Reversible

Part I: Students’ interpretations are dominated by algebra

The students we classified as ongoing reversible considered it necessary to find the algebraic representation before sketching the graph. Dina first involves partial analysis, namely the graph of f' for the interval (-∞, 0) symmetry to the Y-axis, so that the highest degrees are even to the power, i.e. x^2, x^4, and so on. Likewise, with the graph f' for the interval (0, ∞) symmetry with respect to x = 1, so that the highest degrees of the power are odd, namely x^3, x^5, and so on. The same idea is also shown by Nizi, where he states that the graph f' 'is a combination of two functions, namely the quadratic graph for intervals (-∞, 0) and the cubic graph for intervals (0, ∞), and the graph characterizes the “piecewise” function. Both of them tried to find the formula for the function f' that resulted in the graph of f', followed by an integration process to obtain the formula for the function f, and sketched the graph of f. The discontinuity f' at x = 0 causes the two students to consider the strategy of finding the function. The following transcript contains a verbal explanation of the two students’ interpretations of f graph sketches involving derivative charts (see Table 6).

Table 6

The transcript contains Dian and Nizi’s verbal explanations

Think Aloud	*Description*
Dian: *We are asked to sketch the graph f ', so usually if you want to draw a graphic you have to know the function and you can plot it. now I see there are two forms, the first is in the form of a parabola, where for this parabolic shape it touches one point on the X-axis, which is at point (-1,0), now if the form of a parabola like this is, mmm, in my mind I think that this one will have an even rank, it could be a power of 2, like for example x2, x4, and so on. Then for this second one, there is a graph that cuts exactly one on the X-axis at point (1,0), but the shape is not a parabola, now if it's a form like this, I think there will be a function with an odd power, maybe can be x^3, x^5 or so on*	• Interpret the graph f' based on the graphical form • Look for the function of each form
Nizi: *I think that f' is a combination of two functions, a quadratic function equal and a cubic function. This quadratic function has the maximum point at (−1,0). the cubic function has a turning point at (1,0) but its derivative function has an asymptote. as it never touched 0*	• Interpret the graph f' based on the graphical shape

Nizi consistently carried out this thinking for the second problem. He tried to find an algebraic representation that fulfilled the four properties and tried to sketch a graph of each of the known properties. Meanwhile, Dina began to minimise the influence of algebra to sketch graphs, where she perceived that her algebraic representation could be modelled. If $f'(3)$ and $f'(7)$ are given or $f^{'}(3) = f^{'}(7) = 0$. Because both students realised the complexity of the problem by looking for an algebraic representation, they did not continue the solving process. The following transcript contains a verbal explanation of the two students' interpretations of f graph sketches involving analytic properties (see Table 7).

Table 7

The transcript contains Dian and Nizi's verbal explanations

Think Aloud	*Description*
Dina: *this* $f'(1) = 0$ *and* $f'(5) = 0$. *it means that the function is* $(x-1)(x-5) = 0$. *We see that* $f'(x) > 0$ *are at x=-2, x= 1, x=5, and x=7. f'x> 0 at x=3 and x=7 [S3 is silent] how come? I'm confused. This should be* $f'(3) = f'(7) = 0$ *to find the function, which means you can't find the function like that.*	• Looks for a function that satisfies all four properties.
Nizi: *I had thought about looking for its function, but it seems difficult because I have to adjust all the characteristics Then I also wanted to draw the derivative first, but I was afraid it would be more confusing*	• Looks for a function that satisfies all four properties • Sketch the graph.

Verbal expressions on the transcript provide evidence of reversible reasoning that has not been seen when the two students interpret the problem. Their thinking is dominated by algebraic representations and influenced by their familiarity with solving problems that involve functions to sketch graphs.

Part II: Anticipation causes a change in the direction of thinking

The two students could not determine the functional formula of the graph sketch problem involving derivative graphs, so they re-analyzed. With unbalanced conditions, Dina traced back her knowledge of properties of the derivatives, namely, stationary points, monotony, and concavity. She began to realise the relationship between the sign of the first derivative with the behaviour of f, when she stated that "if $f^{'}(x) > 0$, then the graph of f goes up, if $f^{'}(x) < 0$, then the graph of f goes down, and if $f^{'}(x) = 0$, then the graph of f is stationary at point x or f has a local maximum/minimum value ".

This is the starting point for Dina to get the actual f graph sketch (see Figure 5).

Figure 5

Sketches made by Dina

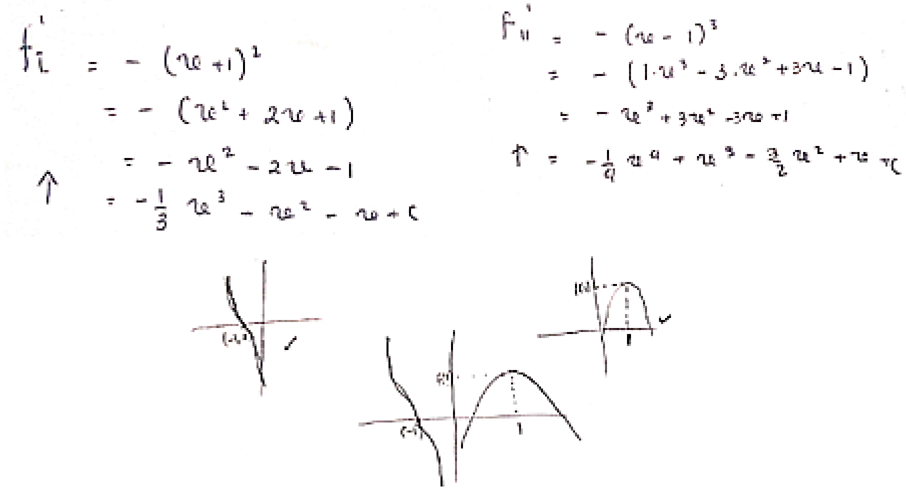
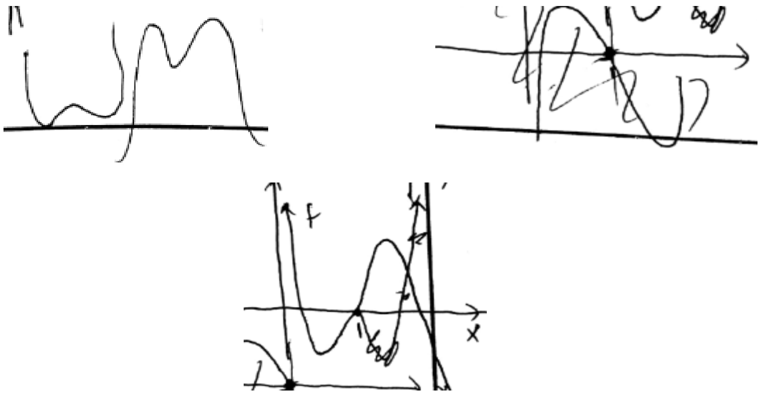


Figure 6

Sketches made by Nizi



Nizi felt the same when she experienced a disequilibrium and realised that utilising the behaviour of functions is not enough to sketch a graph of f.

She then reverses the problem situation, namely looking for conditions at f which cause f' to increase for the interval (-∞, -1) and decrease for the interval (-1,0). In addition, Nizi recalled the concept of derivatives that had been studied before until he realised that the behaviour of f' is related to the concavity of f that when f' rises, f' faces upward (concave up) and when f'decreases, f'faces downward (concave down). Furthermore, she found that $f'(x) > 0$ at the interval (-∞, 0) and (1, ∞) means that f increases in that interval and $f'(x) < 0$ in the interval (0,1) means that f decreases in that interval. She got the sketch of the graph f shown in Figure 6.

The following transcript contains the verbal explanation of the anticipation of the two students when sketching f graph involving a derivative graph (see Table 8)

Table 8

The transcript contains Dian and Nizi's verbal explanations

Think Aloud	*Description*
Dina: *··· one moment. if the f' is greater than 0, it means there is a relationship with the value of x, right? f' is greater than 0, then the function will increase, stationary points occur when f' is equal to 0. oh! this means that if f' is greater than 0 then the function f increases, if f' is less than 0, the function decreases, and if f' is equal to 0 it is stationary, meaning that it has a local maxima/ minima.. ···*	• Gathering her knowledge on the properties of derivatives.
Nizi: *this, I kind of doubt whether it goes up or down. It means that the value of f '(x) is positive, the graph should go up, in the interval of -∞ to 0 and from x= 1 to x towards infinity (∞) . It means that if f' is less than 0 or the graph is below the X-axis, then from x=0 to x=1, f' increases means that f' is positive and the curve is above the axis -X*	• Gathering her knowledge on the properties of derivatives.

However, the structuring of the problems generated by the two students was incomplete. This is because they have not analysed the behaviour of f' at x = 0. Therefore, the researcher conducted a search to clarify- the knowledge of the two through the following interview (See Table 9).

From the interview excerpt, it seems that the two students could not develop their ideas about the relationship between continuity and differentiability. This shows that both students' knowledge of the relationship between derivation and continuity has not been constructed thoroughly but has been used to generalise the problem situation.

Table 9

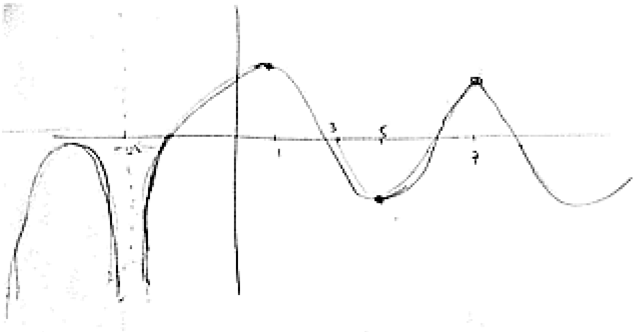
Interview between researcher and participants

Interview	Description
Interviewer: How about a chart f'at x = 0?	
Dina: *It seems I am a little confused because I must find f at x = 0 so it is not continuous at x = 0. Wait a minute, may I remember about the conditions for a continuous function? If there are three limits at one point, then f (a) exists, both whose values are the same. It's a continuous function. But a function is said to be continuous with respect to derivatives. There's no derivative for f, let alone f '= 0, whereas if f is continuous it must have a value of f (a) and it must be the same as this, now there is no limit. I don't think so, therefore, f is not continuous*	• Trace her knowledge about continuity and try to integrate it with the concept of derivatives.
Nizi: *this might be the asymptote; it does not intercept nor touch the curve. I'm confused, it makes me think it doesn't seem like only if there is an asymptote for example. If there is an asymptote as well, I think it's not always that it could be that he is connected*	• Suppose the graph of f is not continuous at x = 0

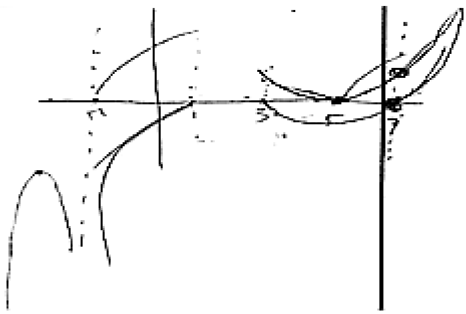
For the graph sketching problem involving derivatives properties, the two students focused on the meaning of the limit function notation $\lim_{x \to -2} f(x) = -\infty$ and $\lim_{x \to -\infty 0} f(x) = -\infty$This resulted in both having to trace back their knowledge of the derivative. Suppose Dina illustrates that f $\lim_{x \to --\infty} f(x) = -\infty$ represents a curve f which decreases as the x value decreases and $\lim_{x \to --2} f(x) = -\infty$ represents the curve f decreases as x approaches -2. Meanwhile, Nizi must return to the basic concept of limit ($\lim_{x \to c} f(x) = L$)) to conclude that the line x = -2 is the vertical asymptote of curve f and curve f decreases as it approaches -∞. Nizi then realises that f reaches its maximum value at a certain point. The sketch results of the two students are shown in Figure 7 as follows.

Figure 7

Sketches made by Nizi



(a) Sketches made by Dina



(b) Sketches made by Nizi

DISCUSSIONS

Reversible reasoning in mathematics education

In mathematics education literature, reversible reasoning is important to develop students in solving problems (Hackenberg, 2010; Ramful, 2014; Simon et al., 2016), because most students need to reason the situation in reverse and involve a deeper understanding of a problem. Reversible reasoning is not limited to numerical aspects only (Ramful, 2015), but it can also be viewed as part of a mathematical connection because students build bi-directional relationships between concepts (García-García & Dolores-Flores, 2019). However, we know little about the characteristics that students develop when they do reversible reasoning.

Ikram et al. (2020) reported that when students sketch original charts based on derivative graphs, they generate two types of reasoning: direct reasoning and reversible reasoning. Furthermore, specifically for reversible reasoning, they seem to show different models when finding the original graphic sketch. Our results are consistent with that students tend to show different characteristics when sketching original charts based on the derivative graph and based on the derivative properties.

Two characteristics of reversible reasoning

The results of this study are in line with Natsheh and Karsenty (2014), Hong and Thomas (2015), Haciomeroglu et al. (2010), and Garcia-Garcia et al. (2019), who explored the graphical understanding of derivatives problems. This graphic problem is a tool for developing reversible reasoning as evidenced by students' answers in this study. However, when we give graphic assignments, they tend to find the function to sketch the graph. We have explored this problem from previous studies that can be seen in Ikram et al. (2020). In this regard, the results we find show that some students exhibit different characteristics of reversible reasoning in establishing the relationship between derivatives and antiderivatives.

Initial Reversible

Initial Reversible occurs when students engage in the accommodation process when analysing a problem. In that sense, the subject's mind is full of questions, thus triggering reversible reasoning. This is due to students' unfamiliarity with the problem situation. This condition stimulates the subject to build new insights, so it takes a long time to understand the problem (Hackenberg & Lee, 2015). In addition, students adapt known procedures to new situations (Maciejewski & Star, 2016). In addition, the initial reversible occurs when students try to interpret each part of the problem. This is due to the maximum effort in interpreting each element of the problem by thinking backwards. When students analyse the causes of discontinuity on the graph f', f' is positive/negative for an interval, and a geometric interpretation $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to -2} f(x) = -\infty$, they can bind new experiences with previous knowledge, assimilate new information, and restructure their knowledge so that it can be used in new situations, so that a well-connected schema is formed (Hodnik Čadež & Manfreda Kolar, 2015; Steele & Johanning, 2004).

Furthermore, the problem formulation affects the students' perspective. In this case, information about the derivative causes the subject to become aware of the initial situation, i.e., graph f that leads to graph f'; and graph f , which satisfied the four given properties. Students perceive problems resulting from the initial situation, leading to a reconstruction of the structural relationship between the initial situation and the results. This is in line with the findings of Ramful (2014) that there is a transformation from result to source through the same path from source to result. Students also develop progress and backward processes so that they view the two situations as an alternating process (Gray & Tall, 1994).

The anticipation model that students generate from these characteristics is as follows. First, they decompose the problem into sub-sections. This shows that the students' decomposition is categorised as an ordered-structural decomposition, where the problem separation into sub-groups is based on a sequence and is continued by analysing each subsection separately (Rich et al., 2019). Second, they trace the properties of the derivative resulting in a graph of f'or the four given properties. This is in line with Krutetskii's (Haciomeroglu et al., 2009) reversibility framework that the subject constructs two-way relationships, namely, from the initial situation to the outcome and from the result to the initial situation. Another indication is the relationship that the subject builds due to their proportional knowledge (Sevimli, 2018) so that they can do reversible reasoning.

Initial Reversible has the same characteristics as the findings of previous studies, where the subject finds the equivalent condition and shows reversible action on activation by dismantling the problem from the beginning to draw conclusions (Ramful, 2014). Simon et al.'s (2016) findings show that reversible reasoning is seen as the result of reflective abstraction, so this characteristic occurs when the internalization of action becomes a process.

Ongoing Reversible

Ongoing Reversible occurs when students transfer ideas from the results of previous experiences. They are accustomed to constructing graphs from a given function, for example, looking for functions that result in the graph f'(Task 1) and the four given properties (Task 2). This is because the subject's mental perception is influenced by problems that have been solved before

(Mcgowen & Tall, 2010), where they are used to constructing graphs based on algebraic expression or given function. Furthermore, students' minds are influenced by symbol-sense which causes any information on the assignment to be expressed through symbols or algebraic forms (Pirie & Kieren, 1994). Then, students do not interpret the elements of the problem analytically, so they experience retroactive interference (Bishop et al., 2014), where the knowledge of antiderivatives interferes with the withdrawal of information about those related to the derivative.

The interpretation, dominated by the assimilation process, causes students to involve direct thinking processes to solve problems (Ikram, Purwanto, Nengah Parta, et al., 2020; Ikram, et al., 2020c, 2020a, 2020b; Ma'rufi et al., 2020; Rahayuningsih et al., 2021). In that sense, students involve an integration process to obtain a graph f. Working with direct thinking makes students bump into a situation and unable to continue their work. This causes a change in the direction of thinking from the direct direction to the opposite. In that sense, when faced with complex situations, students tend to reduce complexity by changing the direction of their thinking to complete their work (Hackenberg & Lee, 2015).

Changing the direction of thinking, from looking for functional formulas to applying derivatives properties, is caused by: encountering complex situations; inability to continue the work based on the initial idea built; and looking for other alternatives by minimising the influence of algebra. This condition resulted in the deconstruction of the workflow done by the students. In the sense that there is a change in the direction of thinking from looking for the numerical behavior of the function to using derivatives properties. This is in line with Simon et al. (2016), who state that to return to the initial situation, a person needs to reconstruct the direction of his thinking by building new knowledge. Changing the direction of thinking occurs when the subject deconstructs an object so that it gives rise to the idea of thinking backwards (W. Kang, 2015), so that they begin to activate reversible schemes in solving problems.

In changing the direction of thinking, students tend to gather their knowledge by recalling their understanding of derivatives and antiderivatives. This is due to a structure gap in the of students' thinking so that they remember existing knowledge vaguely, for example: continuity; the relationship between differentiation and continuity to confirm the behavior f at x = 0 (Task 1); and the definition of the limit function to interpret f $\lim_{x\to-\infty} f(x) = -\infty$ and

$\lim_{x \to -2} f(x) = -\infty$ (Task 2). This indicates that the students recall the understanding needed to continue their work (Pirie & Kieren, 1994). In addition, knowledge about derivatives properties was not formed completely, causing the subject to work in back-and-forth. This is in line with the finding (Tzur, 2007) that students do not immediately realise the reversible scheme when they face a problem, so it takes extra effort to build the scheme.

Other characteristics of Ongoing Reversible including students' thinking processes are dominated by algebraic processes. The flow of thinking shown by the subject is in line with the flow of thinking found by Hong and Thomas (2015). It can be seen when subjects start sketching f based on the given f' graph, namely: assuming a polynomial graph; match it with the function; carry out an integration process to obtain the formula f; and sketching the graph. In addition, students are sometimes unable to describe derivatives properties, especially in the relationship between differentiability and continuity. When viewed from the second finding, the subject was unable to describe the derivatives characteristics, especially on the relationship between derivatives and continuity. The subject's attention is focused on knowledge of the derivatives properties procedurally rather than conceptually, resulting in a missing element in problem structuring (Ron et al., 2017). The incomplete relationship between derivatives and continuity caused the subject to transfer some of the information and generalise it.

CONCLUSIONS

Our research question is "*What are the reversible reasoning characterisations when students sketch graphs of functions involving derivatives*?". Through our analysis, we found two characteristics: Initial Reversible and Ongoing Reversible. The difference between both reasoning characteristics can be seen in how they interpret the problem. The initial reversible is influenced by the visual aspect and is dominated by the accommodation process. In turn, ongoing reversible is influenced by algebraic aspects, so they try to change the problem into algebraic form. These two characteristics may be generally acquired in the classroom. To support students in building these two characteristics, teachers and curriculum developers can consider giving students the opportunity to think with these two characteristics. We hope that our findings raise practitioners' awareness of the diversity of students' reasoning in interpreting graphs.

Our research has direct implications for calculus teaching. In the case of the relationship between derivatives and antiderivatives, it may not be sufficient from an operational point of view. However, it is necessary to pay attention to interrelated conceptual aspects. For example, the relationship between the sign of the first derivative with the increasing/decreasing behavior of f at an interval, the relationship between continuity and differentiability, the meaning of the first derivative, and the meaning of limit notation graphically. Our results also have an impact on the importance of teaching graphic sketches in the curriculum. This is because the various interpretations generated by students have resulted in their visual analysis. Moreover, problem models that make them reduce algebraic thinking have significantly influenced their way of thinking.

Limitations and future researches

We expect that the results of this study motivates lecturers and researchers to design tasks that stimulate students to do reversible reasoning. However, we also believe that to achieve this, students must view the problem by reversing the situation. This has not fully happened, considering that in the teaching and learning process they rarely get a model of the questions developed in this study.

In this study, we make theoretical and methodological contributions. For example, by using the grounded theory method, we produce two characteristics of reversible reasoning. However, future research can improve our findings, for example, by identifying other reversible reasoning characteristics for different problems (e.g. functions and inverses, exponents and logarithms). The two characteristics we find from the data will be important for studying the reversible reasoning that students generate.

In this study, we tried to characterise the reversible reasoning of students when sketching parent function based on a derivative graph and their analytic properties, but we only inferred their mental activity through verbal expressions and interviews, so this is the limit. In future research, we will need to develop different instruments to study their thinking structures or the schemes that have been formed so that they involve reversible reasoning. This is important because the literature describing this issue has not been studied in depth.

AUTHORS' CONTRIBUTIONS STATEMENTS

All the authors have made substantive contributions to the article and assume full responsibility for its content. All those who have made substantive contributions to the article have been named as authors.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author, Mr. M.Marufi, upon reasonable request.

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