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Preservice Teachers' Mathematical Knowledge about Repeating Patterns and their Ability to Notice Preschoolers Algebraic Thinking

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ABSTRACT

Background: Several studies have shown that many preservice teachers (PTs) who teach in the early years have a superficial knowledge about repeating patterns (RPs), which affects their knowledge about children's algebraic thinking. **Objective**: This article aims to understand PTs' algebraic thinking and their ability to notice preschoolers algebraic thinking and how these two domains articulate within a teacher education experiment. **Design**: The study follows a qualitative methodology based on participant observation, complemented by document collection. **Setting and participants:** The study stems from a teaching experiment carried out in a school module focused on patterns and algebra of a degree in basic education, with two pairs of PTs as participants. **Data collection and analysis:** The data come from the written productions and discussions between the elements of each pair of PTs within the scope of the tasks proposed in the teacher education course, adopting an original analysis framework. **Results**: The results reveal that the PTs successfully identify the structure of the RPs and the general position of each term; however, one of the pairs still find difficulties in fully understanding that mathematical object. The pairs attend to relevant aspects of children's algebraic thinking, although sometimes with limited interpretation. **Conclusions:** This study highlights the importance of creating opportunities in initial teacher education for PTs to develop their algebraic thinking from an early algebra perspective and to analyse, in this context, the preschoolers' work.

Keywords: initial teacher education; algebraic thinking; noticing; repeating patterns; childhood education

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O conhecimento matemático de futuras educadoras e professoras sobre sequências repetitivas e a capacidade de *perceber* o pensamento algébrico de crianças do jardim de infância

RESUMO

Contexto: Diversos estudos têm mostrado que muitos futuros educadores e professores (FEPs) dos anos iniciais possuem um conhecimento superficial das sequências repetitivas (SRs), o que afeta o seu conhecimento sobre o pensamento algébrico das crianças. **Objetivo**: Este artigo tem como principal objetivo compreender o pensamento algébrico de FEPs e a sua capacidade de perceber (*noticing*) o pensamento algébrico de crianças do jardim de infância, bem como a articulação entre esses dois domínios, no contexto de uma experiência de formação. **Design**: O estudo segue uma metodologia qualitativa, assente na observação participante, complementada com recolha documental. **Ambiente e participantes:** O estudo decorre de uma experiência de formação realizada numa unidade curricular de Padrões e Álgebra de uma Licenciatura em Educação Básica e tem como participantes dois pares de FEPs. **Coleta e análise de dados:** Os dados provêm das produções escritas e discussões entre os elementos de cada par de formandas no âmbito de tarefas de formação, adotando um framework de análise original. **Resultados**: Evidencia-se que as FEPs identificam com sucesso a estrutura das SRs, bem como a posição geral de cada termo, contudo, um dos pares ainda indicia dificuldades na compreensão plena do objeto matemático. Os pares atendem a aspetos relevantes do pensamento algébrico das crianças, ainda que, por vezes, com limitações na sua interpretação. **Conclusões:** Este estudo realça a importância de criar, na formação inicial, oportunidades para os FEPs desenvolverem o pensamento algébrico numa perspetiva de *Early Algebra* e de analisarem, neste âmbito, o trabalho de alunos dos anos iniciais.

Palavras-chave: formação inicial de FEPs; pensamento algébrico; *noticing*; sequências repetitivas; educação de infância.

INTRODUCTION

The literature has widely discussed introducing algebraic thinking in the early years. The notion of *Early Algebra* – defined as a curricular proposal that consists of integrating algebraic thinking modes in the early years of basic education (Carraher & Schliemann, 2007) – has been highlighted due to the students' difficulties in learning the subject. Algebraic thinking in the early years involves analysing relationships between quantities, developing awareness of numerical structures and properties, studying functional relationships, generalisation and justification, and solving problems focused on relationships (Kieran et al., 2016). As a context for the development of algebraic thinking in the early years, especially functional thinking, patterns stand out, namely repeating patterns (RPs) (Clements & Sarama, 2009; Threfall, 1999; Tirosh et al., 2019).

The literature highlights the need for studies that investigate preservice teachers' [1] (PTs) mathematical knowledge (MK) about the RPs and the knowledge they have about children's knowledge in this area (Tirosh et al., 2019, Tsamir et al., 2019). Thus, teachers and PTs must notice children's thinking about RPs and, for this, they must identify the strategies they use and how they perceive the structure of the sequence and generalise it, besides interpreting these aspects to understand the development of children's in that aspect (Tirosh et al., 2019). Several studies have shown the interrelationship between knowledge about children's thinking and the MK, pointing out that, by improving the math component, teachers and PTs become more efficient in recognising children's learning processes in this subject (Branco, 2013; Tsamir et al., 2019).

Noticing children's thinking is a specific domain of the noticing ability (Jacobs et al., 2010). Although this skill does not have a unique characterisation, it seems to be globally recognised that it refers to attending to important moments, reasoning about them and deciding how to act (van Es et al., 2017). Research has highlighted the importance of this ability for teacher practices (Jacobs et al., 2018; Mason, 2002) and the relevance and need for studies that analyse the noticing of the PTs of the early years in specific mathematical domains, and specifically in algebra context (El Mouhayar, 2019; Llinares, 2019; Walkoe et al., 2020).

In this study, given the particularisation of noticing ability of students' thinking and the characterisation by Sherin and van Es (2009), we understand noticing as the teacher's ability to describe and interpret children's algebraic thinking.

In particular, this article is part of a broader investigation about a teacher education experiment, from an *Early Algebra* perspective, aimed at preeschoolers' PTs. This study aims to understand the PTs' algebraic thinking and their ability to notice preschoolers' algebraic thinking, in the context of RPs. To this end, we sought to answer the following questions: (i) What are the characteristics of PTs' functional thinking in this context? (ii) What are the

[1]In Portugal*, an educator* is an education professional that works at Kindergarten and Preschool levels, *while a teacher* is that who teaches in primary school (children from 6 to 9 years old). However, we will simply use PT to refer to both groups.

characteristics of PTs' ability to describe and interpret preschoolers' algebraic thinking within the scope of the RPs? (iii) How does the PTs' noticing ability relate to their MK about RPs?

THEORETICAL FRAMEWORK

Algebraic thinking and repeating patterns

In recent years, research has shown that early-year students, including preschoolers, are successful in solving tasks related to algebraic thinking and in particular to functional thinking (Blanton et al, 2015; Castro et al., 2017; Oliveira & Mestre, 2014). Blanton and Kaput (2011) conceptualise functional thinking in a broad way, one that incorporates the generalisation of patterns and relationships, through various representational languages and tools and the exploration of generalised relationships or functions, resulting in useful mathematical objects in their own right. Although functional thinking can be developed in various contexts, research has placed particular emphasis on the exploration of sequences or patterns (Carraher et al., 2008; Radford, 2014). In the context of early childhood education and the first years of basic education, the RPs are the most common and those that children come into contact with earlier and can be vehicles for understanding essential relationships, as they allow the discovery of regularities and support children in the development of generalisation skills (Clements & Sarama, 2009; Lynn, 2012; Threlfall, 1999). A RP can be defined as a pattern with a recognised cycle of repeating elements, which is called the unit of repeat (Lynn, 2012; Threlfall, 1999). The unit of repeat, typified as the smallest subset of elements that repeated can generate the sequence (Liljedahl, 2004), is the essential element of the RP. From the unit of repeat and its length (designated as the number of the terms of the unit of repeat), we can determine any term in the sequence.

At an early stage, children can be involved in activities such as continuing and replicating a sequence and identifying missing terms (Clements & Sarama, 2009; Rittle-Johnson et al., 2013; Tirosh et al., 2019). In the case of AB-type sequences (sequences whose unit of repeat consists of two different terms), continuing, replicating, or identifying missing terms are tasks in which the vast majority of preschoolers succeed (Rittle-Johnson et al., 2013; Tirosh et al., 2019). This success is often associated with the use of recursive strategies in which children alternate terms or choose a rhythmic approach, so they must have opportunities to explore the RPs with units of repeat with different lengths and different organisation of elements, to avoid being limited to this type of strategies (Lynn, 2012; Threlfall, 1999). For children to understand the RPs from an *Early Algebra* perspective, it is crucial that the proposed activities provide a path to generalisation, for which the identification of the unit of repeat is important (Lynn, 2012; Threlfall, 1999; Tirosh et al., 2019).

Early years teachers' education in Algebra

Although the role of teachers is essential to promote the development of algebraic thinking in their students (Carraher & Schliemann, 2019), many PTs had little or no contact with *Early Algebra* as students (Hohensee, 2017). Thus, initial training should promote opportunities for this. Carraher and Schliemann (2019) argue for the relevance of the articulation between patterns and functions, stating that it is quite difficult for a teacher to explore patterns with their students without connecting them with functions and relations. As RPs are one of the first contacts of children with activities that promote generalisation (Clements & Sarama, 2009; Tirosh et al., 2019), PTs must get involved in exploring this type of sequences.

In general, studies have shown that, for the most part, first years teachers and PTs successfully carry out a large part of the tasks proposed to children with the RPs. Although most teachers are successful in continuing, replicating, and constructing a RP, knowledge of the mathematical object itself is often quite limited (Lynn, 2012; Tirosh et al., 2019; Waters 2004). Many first years teachers show a superficial perception of RPs, as they resort to recursive strategies, such as children, in addition to manifesting difficulties with the language and the use of specific expressions of this topic (Lynn, 2012; Tirosh et al., 2019; Waters, 2004). In some cases, research has shown that, when involved in interventions focused on RPs, teachers and PTs become more aware of the mathematical aspects associated with this type of sequences, such as the recognition of the sequence structure (from the identification of the unit of repeat), the establishment of relationships between variables, and the creation of rules to determine near and distant terms (Branco, 2013; Tsamir et al., 2019). However, knowledge of the early years PTs is often superficial regarding RPs (Lynn, 2012; Tirosh et al., 2019), so it is necessary to promote a deep understanding of RPs as a mathematical object so that they can support children in the future in the development of algebraic thinking, particularly in the expression of generalisation (Lynn, 2012).

PTs' ability to notice

In recent years, research has focused on the noticing ability of the early years PTs (Callejo & Zapatera, 2017; Jacobs et al., 2010; Walkoe et al., 2020) in the context of initial and in-service education. Although there are different characterisations of this skill, most authors seem to agree that it refers to aspects that are considered in an educational context and highlight two essential components related to the actions of attending and interpreting (Sherin & van Es, 2009; Walkoe et al., 2020). Several authors (Buforn et al., 2020; Jacobs et al., 2010; Llinares, 2019; Walkoe et al., 2020) have focused their attention on a particular aspect of the teacher's noticing ability regarding children's thinking. In short, the teacher's noticing ability regarding children's thinking is defined as the cognitive ability to identify and interpret the outstanding aspects of the children's activity so that, in this way, they can make conscious decisions (Jacobs et al., 2010). To notice children's thinking it is not enough for the teacher to identify the correct and incorrect aspects in the answers; it requires that the teacher evaluates whether they are, or not, significant in the mathematical context and how they can influence the students' understanding of the concepts (El Mouhayar, 2019; van Es et al., 2017). In particular, for teachers to notice children's thinking, it is necessary to reconstruct and make inferences of their understanding based on their productions or interventions (Ivars et al., 2020).

Despite its importance for teaching practices, the noticing ability is not innate to teachers (Ivars et al., 2020; Jacobs et al., 2018), so initial teacher education has been seeking to provide PTs with contact with students' work through means such as the analysis of classroom videos (Rodrigues et al., 2019; Walkoe et al., 2020). Several studies seek to investigate how the MK relates to the PTs ability to notice the children's thinking, and the results have shown that, although the MK is important, it is not enough, especially regarding the interpretation of children's thinking (Buforn et al., 2020; Callejo & Zapatera, 2017; Jacobs et al., 2010; Llinares, 2019). About the noticing capacity in the context of algebraic thinking and, in particular, in the context of sequences, some studies have focused on the development of the components of identification and interpretation of the children's thinking. Besides the perception that these two components are related, there seems to be a consensus that interpretation is a more complex process for PTs (Callejo & Zapatera, 2017; Llinares; 2019; Rodrigues et al., 2019).

METHODS

This study was developed within the scope of a teaching experiment in a school module focused on patterns and algebra offered in the 3rd year of a degree in basic education in Portugal. The teaching experiment, in which the first author took simultaneously on the role of researcher and teacher educator, was held in the 2018/2019 school year, under the theme "From Arithmetics to Algebra: developing the algebraic thinking; Patterns and Functions". The teaching experiment lasted 11 sessions and was designed in collaboration with the teacher responsible for the course. The main objective was to promote the PTs' algebraic thinking and their ability to notice children's algebraic thinking simultaneously, so the tasks with an incidence on MK (MKT) and the tasks targeting the ability to notice children's algebraic thinking (NT) are mostly interrelated. In this way, the MKTs integrate questions that aim to deepen the PTs' algebraic thinking, since the mathematics syllabus when they attended the preschool or primary school as students did not contemplate the early algebra domain. Notably, there was a strong concern in this teacher education course in articulating the notion of repeating pattern as a school topic with the notion of function to make the PTs aware of the nature of this mathematical object (Carraher & Schliemann, 2019). The NTs consist of the analysis of children's written work, transcripts of excerpts from classroom episodes, and videos relating to moments of autonomous work and collective discussion in class, based on mathematical tasks like those solved by the PTs. The tasks were mostly carried out in pairs, with the classes focusing mainly on the PTs' autonomous work, since the course globally adopted an exploratory teaching practice (Canavarro et al., 2014; Hohensee, 2017).

This study follows a qualitative methodology, with the data collection methods centred on participant observation of classes, complemented with audio and video recording, and document collection. In the scope of this article, we selected for analysis the PTs' work concerning two training tasks: *Sequences with children's figures* (MKT – Figure 1), consisting of two parts, and *Sequences with children's figures* (NT – Figure 2).

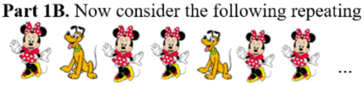
Figure 1

Sequences with children's figures (MKT)

Part 1A. Consider the repeating pattern shown below.

1. What is the unit of repeat?
2. Determine the 14th term of the pattern. Justify.
3. What is the figure in position 355? Justify.
4. Explain how you can determine the position of any figure (*Minnie*, *Mickey*, or *Pluto*) throughout the pattern.

Part 1B. Now consider the following repeating pattern.



1. Indicate the unit of repeat of the pattern.
2. What is the figure in 28th position? And in the 353rd?
3. A part of this pattern has been drawn, starting as shown in the figure, in which the unit of repeat occurs a certain number of times. Indicate and justify if it is possible to have 59 *Minnies in* that pattern.
4. If we consider a part of the pattern in which the unit of repeat appears 32 times, how many pictures in total will be there? And how many *Minnies?* And how many *Plutos?* Explain your thinking.
5. Now using two *Minnies,* one *Mickey* and two *Plutos* build a unit of repeat that origins a pattern with a Mickey in the 639th position. Justify.

Figure 2

Sequences with children's figures (NT)

A group of kindergarten children, aged between 4 and 6 years, solved some tasks about repeating patterns, based on the same figures presented in part 1 of the previous task.

1. The video **Find the unit** shows some situations in which the children had to identify the unit of repeat of those patterns. Compare the four situations, indicating how do you think the particular characteristics of the patterns influenced the response of the different children.

2. In the video **What comes next?** the children try to identify which figure is in a certain position, for some of the patterns displayed.

2.1. Explain how it seems that the children thought about each of the situations.

2.2. Compare the reasoning evidenced by children, in each situation, and try to explain what they reveal about the children's knowledge about repeating patterns.

3. The video **Find out what is missing** shows how the children identified the missing figures when the preschool teacher built a sequence and turned some of the cards down. Compare the two situations presented in this video, referring to the differences between the knowledge that children have to mobilize in each of them.

4. Analyze the video **What is the pattern?**. Comment on the aspects you consider most noteworthy in the episode you have watched.

Both tasks were carried out after the PTs' initial contact with the RPs and the discussion of associated concepts. In the NT, held after the MKT, the

PTs analysed the interventions of the preschoolers (Appendix), aged between four and six, about the RPs similar to those in the MKT.

Two pairs of PTs were selected as participants for this study: Anabela and Bianca and Beatriz and Júlia, who, at the time of data collection, were between 20 and 21 years old. We chose them based on their performance in diagnostic tasks carried out at the beginning of the teaching experiment and the diversity of their paths in secondary school education: Anabela and Bianca successfully attended Mathematics A (the mathematics secondary subject at a higher level), while Beatriz and Júlia did not attend any mathematics subject at that level.

Taking into account the limited research that integrates the domains of algebraic thinking in the scope of the RPs and the capacity of noticing children's algebraic thinking, in this context, we developed a specific framework for data analysis, which is an original contribution of this study. This framework results from the crossing of the two domains considered in the study: (1) MK (functional thinking in the field of the RPs) and (2) ability to notice children's algebraic thinking in the field of the RPs.

Table 1 presents the categories and subcategories of functional thinking analysis in the context of the RPs.

Table 1

Analysis categories related to functional thinking: RPs

Categories	Subcategories
Exploring relationships	Identifying the structure of the sequence Identifying the variables and establishing a relationship between them Establishing a relationship between variables
Generalising	Extending the reasoning from the initial domain Expressing the general rule

Exploring relationships, in the context of RPs, refers to identifying the structure of the sequence, especially the unit of repeat and the cyclic sequential relationship between terms, the variables present and how they relate, i.e., the relationship between order and term (Lynn, 2012; Threlfall, 1999; Tirosh et al., 2019).

In the context of the RPs, generalising refers to the extension of reasoning from the initial domain, through the identification of commonality between cases, and the formulation of a general rule that allows determining any term in the sequence, given its order (Branco, 2013; Liljedahl, 2004).

The data analysis categories related to the ability to notice children's algebraic thinking (Table 2) result from the crossing of the dimensions of functional thinking in the context of the RPs, assumed in Table 1, and the two dimensions of noticing described in the literature, i.e., *describing* and *interpreting* (Callejo & Zapatera, 2017; Jacobs et al., 2010; Rodrigues et al., 2019; Walkoe et al., 2020).

Table 2

Categories of noticing children's algebraic thinking analysis: RPs

			Noticing children's algebraic thinking	
			Describing	**Interpreting**
Functional thinking	Exploring Relationships	Recognising the identification of the unit of repeat	Indicating whether (and how) children identify the unit of repeat	Making inferences about aspects underlying the identification of the unit of repeat and what they reveal about the children's MK
	Exploring relationships and generalising	Recognising the establishment of relationships between terms	Identifying whether (and how) children perceive relationships between RP terms	Making inferences about aspects underlying the determination of terms and what they reveal about children's MK
		Recognising the establishment of relationships between orders and terms	Indicating whether (and how) children determine a term given its order.	Making inferences about aspects underlying the determination of terms given their order and what they reveal about children's MK

In general, *describing* corresponds to the identification of the relevant mathematical aspects present in the students' resolutions/interventions and in their strategies (Jacobs et al., 2010) associated with the recognition of the essential mathematical elements of the students' productions and/or interventions, with retelling and explaining the aspects that attract attention (Estapa et al., 2018, Ivars et al., 2020; Walkoe et al., 2020). For the analysis related to this dimension, the PTs' comments on whether the answers were correct or not are also included (van Es et al., 2017). *Interpreting* refers to the way the PTs reason about the elements they identified and described (Sherin & van Es, 2009; Walkoe et al., 2020), looking beyond what the students wrote or said (Jacobs et al., 2010). This component involves explaining the procedures used, the reasons why an answer is correct or incorrect, and inferring the origin of errors or difficulties (Ivars et al., 2020). This dimension is particularly based on the PTs' inferences about children's algebraic thinking, in which they seek to understand the reasons that led them to present certain written or oral production.

RESULTS

PTs' algebraic thinking

The data analysis presented below is organised by each pair of PTs and focuses on their written productions of the MKT and on the dialogues carried out by each pair when they were solving that task. The MKT presents two length-3 RPs (one of the ABC and the other of the ABA type) with questions that centre on the identification of the respective units of repeat, on the determination of near and distant terms, on the number of figures given a specific number of occurrences and a general rule, and in the construction of a new sequence given specific conditions.

Anabela and Bianca

*Exploring relationships***.** The PTs successfully identify the structure of the two sequences presented, indicating that the unit of repeat of the ABC-type RP is "Minnie, Mickey, Pluto" and that of the ABA-type is "Minnie, Mickey, Minnie".

The pair also identifies the composition of the ABA-type unit of repeat and the number of figures of each type that constitute it. When asked about the possibility of there being a “part” of the sequence with 59 Minnies, the PTs show that they understand that the number of Minnies is always even (provided the unit of repeat is presented completely), as stated by Bianca: “There will never be [59 Minnies] because it’s an odd number. You, in the repetition of the unit, always have an even number of Minnies”. However, in their written production, they resort to proportional reasoning, presenting a rule of three, which seems to indicate that they feel the need to justify the way they think through calculations. The PTs identify, also without difficulty, the number of figures of each type when the unit of repeat occurs 32 times, indicating that there would be 32 Plutos, 64 Minnies, and 96 figures in total.

The PTs identify the length of the unit of repeat, as mentioned by Anabela, in the pair’s discussion: “To know the fourteenth term, we have to divide the fourteen by the three, which is the length”. The pair uses this knowledge to determine the terms of the sequence, showing that they meet a relationship between the variables, even though they do not explicitly identify them. The PTs’ written production (Figure 3) allows us to infer that they establish a relationship between the position (order) of a figure in the sequence and the respective term (type of figure), both for an ABC-type RP (first resolution – Figure 3) and an ABA-type RP (second resolution – Figure 3).

Figure 3

Resolution of Anabela and Bianca from MKT - Q2 (1A) and Q2(1B)

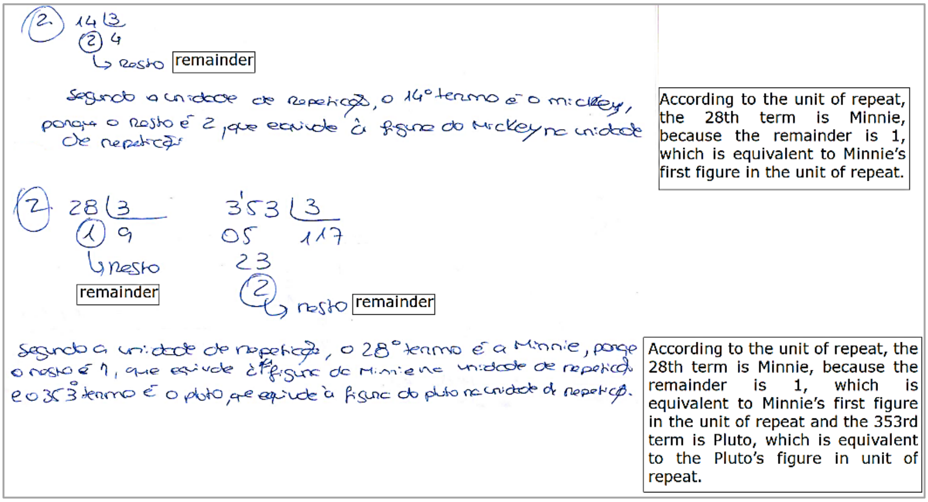


Figure 4

Resolution of Anabela and Bianca of the MKT - Q5(1B)

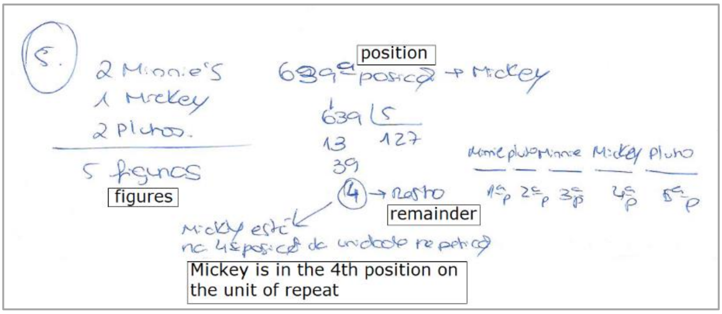
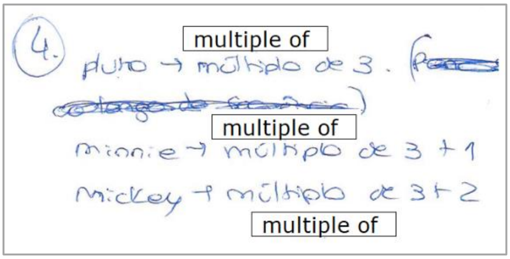


Figure 5

Resolution of Anabela and Bianca of the MKT - Q(1B)



Generalising. From the resolution presented in Figure 3, it is possible to infer that the PTs understand that there is an equality between each term of the RPs and one of the first three terms. In determining the requested terms, they resort to the division algorithm to obtain the value of the remainder. Although their written explanation does not explicitly refer to the position each figure occupies in the unit, the pair's dialogue shows that they unequivocally relate each figure with its position, as expressed by Anabela: "As the remainder is two, it is equivalent to the second term of the unit of repeat". Thus, these PTs show that they understand the correspondence between each term and its order, extending the reasoning beyond the initial domain, and identify a process that allows them to determine any term of an RP through the division algorithm. Based on the fact that they know that by identifying the unit of repeat and its length it is possible to determine any term, the pair succeeds in building the unit of repeat of a new RP that was requested of them (Figure 4).

The PTs' written production and dialogue between them reinforce the conjecture that they identify a general procedure they adopt for all cases, as mentioned by Bianca: "They want a sequence in which Mickey is in this position, by so let's make the rule to see what position Mickey is in the unit of repeat". Their speech, together with their resolution, and specifically the fact of using the word "rule", show that they identify the validity of the relationships that can be established between orders and terms for any RP.

Besides showing that they identify a general relationship between terms and orders in the RPs, PTs are successful in expressing the general rule that relates a term to its order (Figure 5).

In the case of the ABC-type RP, the pair presents a rule in which they identify Pluto as the figure that is always in positions multiple of three, establishing, from there, the general positions of the remaining figures. The fact that they use the addition of the remainder to determine the term of specific order and not subtraction possibly indicates that the PTs are focused on the procedure they consider general, similarly to what they had already shown in the construction of a new sequence (Figure 4).

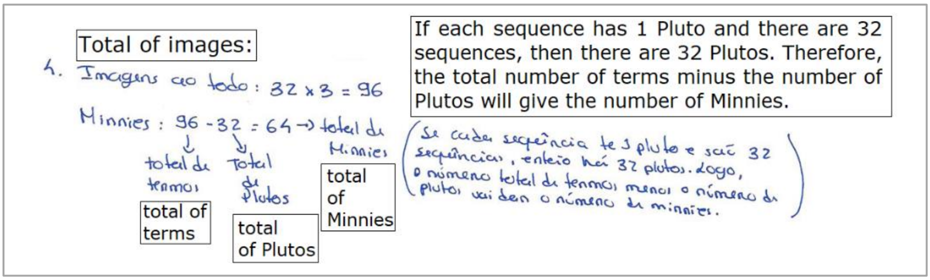
Beatriz and Júlia

Exploring relationships. The PTs identify the structure of the sequences in question, indicating that the unit of repeat of the first RP presented (ABC-type) is "Minnie, Mickey, Pluto" and that of the second (ABA-type) is "Minnie, Mickey, Minnie".

This PT pair never directly mentions the length of the unit of repeat, but the dialogue between them allows us to infer that they identify it, by recognising that, in the case of the ABC-type unit of repeat, the multiple positions of three correspond to the Pluto figure, as mentioned by Beatriz: "Pluto is always a multiple of three, right? Three, six, nine?". Her speech reveals that by continuing the sequence (possibly done without the need to resort to external representations), she identifies a relationship between the terms whose figure is Pluto and their order, establishing a relationship between the variables. The PT pair use this information to determine the 14th term indicating, in their written production, that "Since Pluto is always a multiple of 3, the 14th term will be Mickey, as it comes before the multiple of 3". The analysis of this written production allows us to infer that they understand that the figure before Pluto corresponds to Mickey, which highlights the identification of relationships between terms.

Figure 6

Resolution of Beatriz and Júlia of the MKT - Q4(1B)



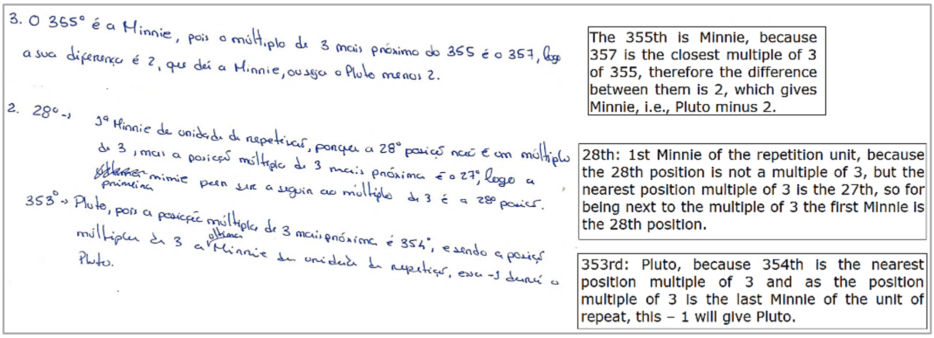
The PTs correctly relate the number of figures of each type to the unit of repeat but show some difficulties in perceiving the cyclical relationship between terms. Initially, when asked about the possibility of a sequence in which the unit of repeat appears a certain number of times to have 59 Minnies, they start by confusing the existing number of Minnies with the order in the sequence, using the calculator to look for the multiple of 3 that is closer to 59. From the conversation between the PTs, it is possible to infer that they initially do not understand the question presented and try to relate the indicated number with the multiplicity of terms based on the relationship they identify for multiples of three. However, they eventually recognise that they were not thinking correctly, and Beatriz assumes that the number of Minnies would always be even: “No, no, each unit of repeat always has two Minnies, so the number of Minnies will always be even”, and Júlia identifies that the number of Minnies only would be odd if the unit of repeat was not presented in its entirety: “If not, the sequence was incomplete”. Although they verify that each unit of repeat has two Minnies, Beatriz and Júlia, again, show difficulties relating the number of terms with the constitution of the unit of repeat when they are proposed to determine the number of figures of each type that appear in the sequence when the unit of repeat occurs 32 times. Even so, the PTs easily identify that the total number of figures is 96 and that the number of Minnies is always double the number of Plutos, evidencing they recognise relationships between terms, but that it is not enough to directly determine the number of Minnies. Thus, they need to resort to the difference between the total of terms and the figures that correspond to Pluto. In their written production (Figure 6), they explain the way they thought, nevertheless that does not allow us to infer whether they have understood the relationship between the number of times the unit of repeat appears and the total number of Minnies.

The fact that PTs write "each sequence" indicates some lack of rigour in the use of specific mathematical expressions. The pair know and use the expression "unit of repeat" and, in resolving this question, apparently assume that it is indifferent to resort to this expression or "sequence", which indicates an unclear notion of the concepts involved.

Generalising. In line with the fact that they have identified that the figure in positions multiple of three always corresponds to the last figure of the unit of repeat, the PTs developed a procedure allowing them to determine any term, given its order. The pair identifies the multiple of three closest to the requested order and move back or forward one or two units to reach the intended order (Figure 7), both for the ABC-type (Question 3 - 1A) and ABA-type RP (Question 2 - 1B).

Figure 7

Resolution of Beatriz and Júlia of the MKT - Q3(1A), Q2(1B)



In their written production, the PTs do not explain the strategy used to determine the closest multiple, however, their dialogue allows us to state that they use the calculator to resort to a strategy of trial and error. For example, in determining the 355th term of the ABC-type RP presented, after discovering that the closest multiple of 3 is 357, Júlia states: "So it's minus two, so it's Minnie, right? (…) because Minnie is the multiples of three minus two". The dialogue between the PTs, especially Júlia's statement, shows that they successfully relate any term to their general position; however, their written production essentially highlights the relationship between terms. In their written work, the PTs identify a general rule that relates terms and positions, but go back to focusing on the relationship between terms, using Pluto as a reference to determine any term: "As we know that the Pluto figure is always a multiple of 3, the other figures will always be -1 or -2 than Pluto or +1 or +2 than the same [Pluto]". Although these PTs are successful in expressing the general rule, they show it is a challenge for them when they are asked to build another RP with a new unit of repeat, given its length and a distant term (Question 5 - 1B), possibly by having been, until then, too focused on the procedure to find distant terms. The PTs are not sure about whether to divide 639 by 4 or by 5, doing several trials to try to build the new RP. Possibly because the question is much more complex than the previous ones and there are some difficulties associated with understanding the essential characteristics of an RP as a mathematical object, the pair finally give up on solving the question without making any record.

Noticing children's algebraic thinking

The data analysis presented below is organised by pairs of PTs and centred on their productions in relation to the NT and on the dialogues each pair carried out when resolving it. This teacher education task features a set of videos from preschoolers in which they explore the RPs. The questions posed to the children focus on the identification of the unit of repeat of the various sequences and the determination of terms.

Anabela and Bianca

Exploring relationships. In the analysis of the video "What is the pattern?" (Appendix), the PTs, when seeing the presented sequence, consider that its length is 6. Based on their perception of the sequence structure, the pair analyse the children's intervention, assuming that their actions result from difficulties in identifying the length of the unit of repeat. In their written production, they state:

> The children say that this is not a pattern because […] they perceive that one of the figures is not in the correct position,

> because they are not able to recognise the unit of repeat with six terms, but only with three terms.

Possibly because they assume that the children's intervention results from the non-identification of a unit of repeat with six terms, the PTs do not address fundamental aspects of this episode, such as the fact that children verbalise the recognition of relationships in relation to the position of the figures in the unit of repeat, which shows that their analysis is highly conditioned by their own perception of the presented sequence.

In the analysis of the video "What comes next" (Appendix), the PTs also use the children's identification of the unit of repeat to justify how they determine the terms of some RPs. In the first situation presented, the pair consider that the children determine a close term by repeating the unit of repeat of type AB and suppose in writing that they "know what a unit of repeat is because to reach the result, they always repeat the unit of repeat". This statement about the "knowledge" of the unit of repeat is, perhaps, a consequence of the fact that, in the video, children apparently resort to a rhythmic strategy in which they repeat the "name" of the figures. In the pair dialogue, Bianca says: "I think it's by the repetition because they're always saying "Minnie, Mickey, Minnie, Mickey". In this situation, they reached the result by repeating the unit of repeat". Her speech shows that she associates the correct cyclical repetition of terms with the identification of the unit of repeat. Thus, the PTs assume that by correctly "reading" the sequence, children identify the unit of repeat, not questioning whether this correct reading results from the perception of the set of terms that repeats itself cyclically or just from a recursive strategy in which the terms alternate.

In the analysis of the third situation, the pair dialogue shows that, based on the children's actions, the PTs assume that they identify each figure's position in the unit of repeat, as Bianca says: "I think they can understand that the second figure in the unit of repeat is always a Pluto. We can say that they arrived at the right result from the unit of repeat, by the position each figure occupies in the unit". Her speech allows us to infer that she recognises that children perceive a relationship between the terms of the sequence and, in particular, the equality between a given term and one of the first three. However, they do not register any of these aspects, just retelling in their written production what the children did, without making inferences about their MK.

Generalising. Anabela and Bianca analyse situation 2 in the video "What comes next?" (Appendix), in particular Maria's intervention, focusing on how the child determines the requested terms. The PTs' dialogues show the understanding that this child possibly thinks in a more abstract way than the others. In particular, the following statements emerge: "[Maria] does not resort to the unit of repeat" and "it seems that she perceives the pattern itself". Although they do not explain exactly what they intend to say, their comment about the unit of repeat indicates that they consider that Maria does not need to follow the unit of repeat term by term, seeing the unit of repeat as a whole. The statement about the perception of the pattern apparently refers to the understanding of the structure of the sequence and general relations. Thus, the pair recognises that Maria extends the relationship between terms from the initial domain to a relationship between any terms in the sequence and also assumes that the child has the notion of regularity in every four terms, as referred by Anabela: "she sees that four out of four will also be the same". In their written production, the PTs seek to identify the strategy used, indicating that "Maria gets the result by doing math/calculations", which possibly refers to the decomposition of eight (the order of the requested term) that she does into two groups of four. They also mention that "Maria is able to identify that before and after Minnie there is always a Mickey", which reinforces the idea that they infer that the child recognises the commonality between cases and generalises it to any case. Throughout the analysis of the children's production, the PTs resort to the "pattern" expression apparently synonymous with SR and also as a way of expressing the structure of an RP, when referring to Maria's intervention ["it seems that she perceives the pattern per se"], which allows us to infer that they sometimes do not use RPs' specific expressions with precision.

Beatriz and Júlia

Exploring relationships. When analysing the video "What is the pattern?" (Appendix) these two PTs, just like Anabela and Bianca, assumed that the length of the unit of repeat of the sequence presented is 6. In their written production, the PTs consider that children:

> find it difficult to identify the pattern unit, as it is composed of six terms, but the same term appears three times, but in different positions (…) they cannot see that, in the same pattern unit, the terms can be repeated in different positions.

Since for the pair the unit of repeat consists of six terms with identical non-consecutive figures, they associate the children's difficulties with the complexity of the sequence's structure. Their perception of children's intervention is strongly conditioned by their own MK, essentially focusing on the length of the unit of repeat without considering fundamental aspects of the situation, such as the fact that children recognise relationships between terms and establish relationships between the figures and its position in the unit of repeat.

In the analysis of the video "What comes next?" (Appendix) the PTs essentially meet the strategies used by children in determining terms and compare their MK within the scope of the RPs. In the first situation, Beatriz and Júlia write that children determine the ninth term through repetition, indicating that "they finger-count the terms up to nine (...), revealing a more elementary knowledge". The PTs show that the resource to an active representation (finger-counting) is associated with less developed knowledge than the other interventions. Although they consider important aspects of the children's intervention, the pair does not recognise that the children initially determine the ninth term without finger-counting, showing the establishment of relationships between terms. By comparison, in the analysis of the third situation, the PTs attend to the fact that children do not need to "finger-count" and rely on the composition of the unit of repeat to determine the eighth term. In particular, in the pair discussion, Beatriz says that the children "they see the pattern unit, they know next it would be Mickey, immediately after it was Pluto (…) they know why there would be that [term]", which indicates that she considers that the children establish a relationship between the figure and its position in the unit of repeat, which, in their written production, they associate with a "knowledge of pattern sequences (...) more developed than 1[Situation 1]". The PTs use expressions such as "pattern unit" or "pattern sequences", which similarly to the MKT resolution highlights some difficulties in the use of expressions associated with the RPs.

Generalising. In the analysis of situation 2 in the video "What comes next?" (Appendix), the conversation between the PTs about Maria's intervention allows us to conclude that they recognise important elements and that they relate the way the child determines the requested terms with an ability to think in a more abstract way than the others. The pair infers that Maria understands that there is a regularity, both in every two terms, as in every four, as indicated by Júlia, when saying that "[Maria] saw that she needed a pair like this one" and, by Beatriz, when indicating that the child "seems to understand that every four will always be one Mickey". Regarding the fact that Maria identifies that between two Minnies there is always a Mickey, Beatriz says: "[...] and when saying that the Mickey is always between two Minnies. (…) she does it straight away by logical thought (…) she thinks straight away, automatically, without doing computations". The excerpt appears to acknowledge that Maria goes beyond the domain of already known terms to indicate a relationship for any two terms. Possibly due to Maria's speech, Júlia assumes that "she already can make a rule" and in the written production, the two PTs emphasise it as they indicate that Maria

> realises that if the term 11 is a Minnie, the 10 and 12 must be a Mickey to respect the pattern unit. This reveals a more developed thinking about repetition sequences than the previous one because it already manages to form a rule.

The speech of the PTs and their written production allows us to conjecture that, for them, the "expression of a rule" means a more developed knowledge in this area.

FINAL CONSIDERATIONS

In this article, based on the analysis presented, we tried to understand the algebraic thinking of two pairs of PTs – Anabela and Bianca and Beatriz and Júlia – regarding functional thinking in the context of the RPs, their ability to notice preschoolers' algebraic thinking, and how the two domains (MK and noticing) are related to each other.

About the MK within the RPs, the two pairs successfully identify the structure of the sequences. Regarding identifying units of repeat of the sequences presented, both pairs do it without difficulty and immediately. Establishing relationships between the number of elements of each type and the unit of repeat is problematic for Beatriz and Júlia, who, although eventually answering the questions associated with these relationships correctly, do not show that they understand them in depth. The two pairs successfully determine any term, given its position in the sequence, establishing a relationship between terms and their respective orders, similarly to the results of the study by Branco (2013). Both show an understanding of the equality between each term of the RPs and one of the first three terms, correctly identifying the general position of each term. Anabela and Bianca focus on the identified relationship and thus their perception about the structure of the RPs allows them to successfully build another sequence with specific characteristics, given the figures that constitute it and their number. Beatriz and Júlia focus on the procedure that allows them, from identifying a multiple of three, to remove or add units until the desired order is obtained. The fact that they focus on the procedure influences their understanding the process of building a new RP, which indicates difficulties in understanding the RPs as a mathematical object, which is in line with the studies by Lynn (2012) and Tirosh (2019). The pair's difficulties in identifying key aspects of the RPs as a function, which prevent them from successfully solving more complex issues, highlight the importance and need to enhance the articulation between sequences and functions (Carraher & Schliemann, 2019) in the PTs' initial training.

Regarding the noticing ability, the two pairs attend to important aspects related to the children's interventions in the videos and, based on them, make inferences about their understanding in the context of the RPs. The two pairs *describe* the children's interventions, retelling the situations presented and considering relevant aspects. Although both analyses include elements of *interpretation* of children's algebraic thinking, in which the pairs try to explain how they have thought, at times, especially in the case of Anabela and Bianca, they eventually focus on retelling the situations. This difficulty in *interpreting* aspects addressed is in line with several studies (Callejo & Zapatera, 2017; Llinares, 2019; Walkoe et al., 2020), which underscore the added difficulty of the interpretive component.

Despite presenting more difficulties with regard to MK in the context of the RPs, Beatriz and Júlia notice the children's interventions in a deeper way and, in particular, they are the only ones that compare the children's performance in different situations and associate the strategies used to the development of MK in the RPs' domain. Although these results are very specific, they contribute to reinforce the conjecture that the MK per se is not enough to notice students' thinking, as also mentioned in the investigations by Callejo and Zapatera (2017) and Jacobs et al. (2010). However, although it is not the only determining factor in the ability to notice children's algebraic thinking, the results show that the MK limits these PTs' analysis. This is particularly evident in their analysis of one of the videos where they do not identify and, consequently, cannot interpret (Llinares, 2019) central elements about children's algebraic thinking because they are conditioned by their own identification of the structure of the sequence. As Lynn (2012) and Waters (2004) studied, difficulties associated with the use of specific mathematical expressions of algebraic thinking and, in particular, of the RPs, are evident in the results, mainly in the case of Beatriz and Júlia, and influence both the resolutions regarding MK and the analysis of children's algebraic thinking.

The results also show that, although the pairs mentioned important aspects in their written productions, in several cases, they show a much deeper perception of the children's interventions in their dialogues than what their written productions reveal. The fact that they are sometimes not successful in registering aspects they take into account should be considered in the promotion of the noticing capacity and the very research on this capacity. Conversations between PTs about students' interventions or productions can allow teacher educators and researchers to access aspects that would not be possible only through written productions and support them more effectively in developing their noticing ability.

By being in contact with the children's interventions, the PTs necessarily deepened aspects of algebraic thinking associated with the RPs, which may have made them more aware of their own MK (Appova & Taylor, 2019). Although this conjecture needs more elements, namely the analysis of the MK of the PTs in later tasks, we infer that taking into account the difficulties of the PTs pointed out in the literature in the scope of the RPs, the integration of the analysis of students' thinking (in this case, of preschoolers) can be an asset in initial training, in courses aimed at the development of algebraic thinking.

This article also developed a specific and original framework for the analysis of noticing capacity related to children's algebraic thinking regarding the RPs. From our viewpoint, the fact that the framework is associated with only one task and specifically with the analysis of preschoolers' interventions in a first contact with the RPs, in which few aspects associated with generalisation emerge, is a limitation for this study. However, we believe that considering the importance of developing the ability to notice children's algebraic thinking in this context, this framework can be an asset for teacher education and research and can be applied or adapted to other contexts and educational levels.

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AUTHORSHIP CONTRIBUTION STATEMENT

The three authors actively participated in the development of the study's theory and methodology, as well as in the discussion of results and conclusions.

DATA SHARING DECLARATION

The data supporting the results of this study will be made available by the corresponding author upon reasonable request.

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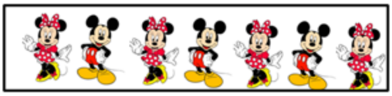
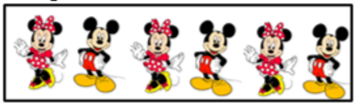
APPENDIX - DESCRIPTION OF THE ANALYSED VIDEOS



Video "What comes next?" This video presents three situations in which some groups of children explore repeating patterns, with focus on determining terms.

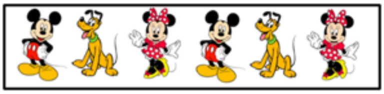
Situation 1 – The children "read" the pattern (on the right) and the preschool teacher asks what's next to Minnie. The children immediately say, together, "Mickey". The preschool teacher then explains that they must associate a number to each card and the children, together, count to seven, pointing to the correspondent card (on the table are seven cards aligned). The preschool teacher asks "So what would number 9 be? Which figure would number 9 be?" and two children immediately respond "Minnie". The preschool teacher says that she "can only see" by counting through her fingers and asks one child to put nine fingers on the table. The preschool teacher points to each finger saying "Minnie, Mickey…" until she reaches the ninth position and says "Minnie, is that what you said?" to which the children respond in chorus "Yes!".



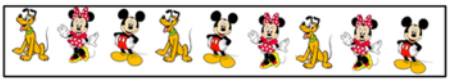
Situation 2 – In another moment, as a request of the preschool teacher, Maria, one of the children from another group, builds a repeating pattern of her choice (pattern presented on the right).

The preschool teacher asks Maria what is the base of the pattern and the child replies "Minnie, Mickey (pause), Minnie, Mickey, (pause), Minnie, Mickey". Then the preschool teacher asks her about the eighth position: "what will be the eight?". As Maria points to each of the figures, it is quite clear that she is associating a number to each card and points to the empty part of the table with gestures that indicate that she is counting the positions missing to number eight. Maria looks at the cards for a while and says "by doing computations I can find it". The preschool teacher asks her what is the number of the last card and she replies “Six”. Then the preschool teacher asks “So what would be the eight?” and the child answers “Mickey”. When asked about how she determined this term, Maria replies: “Four plus four is eight ... we had to put two more”. The preschool teacher congratulates her and says that “if we already had this pair, we had to put another pair, didn't we?” and Maria says “yes”, that she had to put Minnie and Mickey. After this the researcher questions the child about what should be figure 10, knowing that 11 would be Minnie and Maria answers that it would be a Mickey. After the researcher asks her to justify, they have the following conversation:

> Researcher: Why?
>
> Maria: Because Minnie is eleven and Mickey is twelve... No, it’s ten!
>
> Researcher: It’s twelve, but it’s also the ten, why? Between two Minnies there is always...
>
> Maria: A Mickey.



Situation 3 – The repeating pattern on the right is presented to the children. The preschool teacher asks the children how many cards are presented and they reply “six” and she also asks them “what’s the base?”. One child immediately answers “Three” and separates the two units of repeat. Then the preschool teacher asks the children about what would be the “eighth card”. Children point to Pluto which is in fifth position and the preschool teacher confirms, saying that in position eight there is Pluto. The researcher asks children to justify and one of them points to Mickey who is in the fourth position and justifies that "Because seven was Mickey."



Video "What is the pattern?" This video shows a situation where a group of three children is confronted with the repeating pattern on the right. The preschool teacher asks the children to look at the cards and say what is the pattern. Children begin to “read”: “Pluto, Minnie, Mickey, Pluto, Mickey…”. Immediately they look at each other, “suspicious” of the given pattern and have the following dialogue:

> Maria: Here’s swapped, here’s Mickey (Maria and the other two children point to the sixth position, where they believe Mickey should be).
>
> Preschool teacher: Is it? Do the whole reading [of the pattern].
>
> Maria: Because Minnie is the second and now it’s Mickey.
>
> Isabel: Minnie must be next to Pluto.
>
> Maria: This is Minnie (points to fifth position).
>
> Preschool teacher: Is it? It might not be...
>
> Maria: To be a pattern here (points to the fifth position) it had to be Minnie.