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Didactic-Mathematical Knowledge and Teacher Education: An Investigation with Pre-Service and In-Service Mathematics Teachers

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ABSTRACT

Background: The mathematical and didactic knowledge in mathematics teachers' domain should constitute a permanent focus of investigations and reflections. **Objective**: To investigate the development of knowledge and procedures concerning geometry teaching among pre-service and in-service mathematics teachers engaged in the final years of elementary school. **Design**: The investigation is inserted in a qualitative perspective and follows the assumptions of action research. **Setting and Participants**: Fifty-four pre-service and in-service mathematics teachers from the Brazilian city of Paulo Afonso, Bahia, participated in a teacher education process. **Data Collection and Analysis**: Participant observation, data collection and analysis following an analysis protocol guided by the constructs of the onto-semiotic approach; analysis of productions. **Results**: Development and application of 24 sequences of activities focusing on the study of geometry. High epistemic adequacy was observed based on the application of the analysis protocol concerning structuring sequences of activities. Regarding the other dimensions, concerning the analysis of the application of the activities (cognitive-affective, interactional-mediational, ecological), high adequacy was observed. **Conclusions**: It is relevant to invest in teacher education courses proposing the expansion of didactic-mathematical knowledge to enhance the use of different ways to teach and learn mathematics.

Keywords: Teachers' knowings; Teachers' knowledge; Didactic-mathematical knowledge; Mathematics education.

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Conhecimentos didático-matemáticos e a formação de professores: uma investigação com professores e futuros professores de matemática

RESUMO

Contexto: Os conhecimentos matemáticos e didáticos que são de domínio dos professores de Matemática devem se constituir em foco permanente de investigações e reflexões. **Objetivo**: Investigar o desenvolvimento de conhecimentos e procedimentos referentes ao processo de ensino da Geometria junto à professores e futuros professores de Matemática dos Anos Finais do Ensino Fundamental. **Design**: A investigação se insere em uma perspectiva qualitativa e segue os pressupostos de uma pesquisa-ação. **Cenário e Participantes**: 54 professores e futuros professores de Matemática do Município brasileiro de Paulo Afonso/BA, participantes de um processo de formação de professores. **Coleta e Análises de Dados**: Observação participante, coleta e análises de dados seguindo um protocolo de análise orientado pelos constructos do Enfoque Ontossemiótico; análise de produções. **Resultados**: Desenvolvimento e aplicação de 24 Sequências de Atividades com foco no estudo da Geometria. Com base na aplicação do protocolo de análise, referente a estruturação das Sequências de Atividades, observou-se alta adequação epistêmica. Já com relação às demais dimensões, referentes à análise da aplicação das atividades (Cognitivo-Afetivo, Interacional-Mediacional, Ecológica), observou-se uma alta adequação. **Conclusões**: É relevante o investimento em curso de formação de professor que proponha a ampliação de conhecimentos didático-matemáticos que potencializem as ações de se utilizar de distintas maneiras de se ensinar e aprender Matemática.

Palavras-chave: Saberes docentes; Conhecimentos docentes; Conhecimento Didático-Matemático; Educação Matemática.

INTRODUCTION

The reflections presented in this article emerged from the context of studies carried out to compose a doctoral thesis that aimed to investigate the development of knowledge and procedures related to the teaching process of geometry with pre and in-service mathematics teachers engaged in the final years of elementary school in the city of Paulo Afonso, Bahia, based on a training process structured from the perspective of the onto-semiotic approach to knowledge and mathematical instruction (OSA). Thus, we proposed a mathematics teacher education course to investigate the mobilisation of aspects of didactic-mathematical knowledge highlighted in the onto-semiotic approach in the context of a formative process.

To meet the research objectives but, fundamentally, structure the training process, we began our study based on Gauthier's theoretical constructs *et al.* (2013*)* and Tardif (2002, 2014)to understand the teacher's teaching knowings, going on with reflections that encompass the knowings mobilised by specialist teachers in specific subjects with arguments based on Schulman (1986, 1987), reaching the discussions proposed by Ball, Thames, and Phelps (2008) and Hill, Ball, and Schilling (2008), who, in line with the preceding ideas, presented a model of teaching knowings that the mathematics teacher should mobilise. This path allowed placing the theoretical constructs proposed by the onto-semiotic approach to knowledge and mathematical instruction (OSA) as the basis for conducting teacher education and for the analyses produced in the scope of the investigation (Godino, Batanero & Font, 2008; Godino, 2009; Pino-Fan & Godino, 2015; Godino *et al*., 2013; Godino *et al*., 2017).

Thus, within the scope of the OSA, the didactic-mathematical knowledge model (DMK) was taken as a reference, and within the framework of the model, the use of the didactic suitability criteria (Pino-Fan & Godino, 2015; Godino *et al*., 2017) for the development and analysis of the teacher's didactic-mathematical knowledge (and competencies). In this sense, the DMK constructs make it possible to generate arguments that consolidate the influences of these analysis resources in enabling the teacher to overcome the limitations associated with teaching through strategies and methodologies that meet the students' perspectives and propose reflection opportunities and analyses of the very pedagogical practice.

In line with theoretical assumptions, the investigation is part of a qualitative perspective along the lines of action research (Prodanov & Freitas, 2013). We analysed the didactic-mathematical knowledge (DMK) mobilised by pre-service and in-service teachers about the structuring and application of sequences of geometry activities, considering the components and indicators proposed by the didactic suitability analysis guides[1]- mathematics (DSAGM) of the OSA, proposed by Godino *et al.* (2013).

Next, we present theoretical notes taken as references in the study, following the path for its constitution, as already highlighted.

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[1]Didactic Suitability is a "[...] system of empirical indicators found in each of the facets that guides the analysis and systematic reflection that contributes as criteria for the progressive improvement of teaching and learning processes" (GODINO *et al.*, 2017, p. 95).

TEACHING KNOWLEDGE AND KNOWINGS

Seen as a continuous process of growth and development, education requires that school – as an institution- and the teachers who work in it be inserted considering the changes before paradigms that are no longer sufficient or suitable to the demands arising from the current needs. In these times, the teacher's "[...] education assumes a role that goes beyond teaching that aims at a mere scientific, pedagogical, and didactic update and becomes the possibility of creating spaces for participation, reflection, and education" (Imbernón, 2011, p. 19). Referring to such professional demand, Zabala (2002, p. 57) states that it is about "[...] educating for innovation people capable of evolving, adapting to a rapidly changing world, and mastering change", highlighting the importance of developing an attitude of permanent education, guided by the skills of learning to learn and work in teams.

Taking these assumptions into account, one can see the relevance of teacher education in the context of providing feedback on the teaching knowings, in the sense that the process of constituting a teacher's professional identity begins in the space of initial education but, to solidify it, there must be spaces for experiences, exchanges of knowings with other teachers and theoretical reflections through studies and different experiments in the classroom.

Thus, the studies point out that the understanding of teaching is established through peculiar knowings, characteristic of the teaching profession. However, for a long time, the teachers' teaching was based on their notions, on their understandings. People believed that, to teach, it was enough to "know the content, have talent, have common sense, follow intuition, have experience, have culture" (Gauthier *et al.*, 2013, p. 20), pointing to the difficulty of defining the knowings involved in this craft.

In this context of formalising the teaching profession, which Gauthier *et al.* (2013) call a "craft made of knowings", a set of knowledge is presented that the teacher must mobilise for teaching, such as, for example, management of lesson plans, teaching methodologies, control of class discipline/indiscipline, school assessment, content knowledge, student learning stages, besides being points of reflection" [...] were validated by research and should be incorporated into teacher education programmes" (Tardif, 2002, p. 1).

Given this scenario, teaching is now evaluated "[...] as the mobilisation of various types of knowings that form a kind of reservoir in which the teachers supply themselves to respond to the specific requirement of a concrete teaching situation" (Gauthier *et al., 2013, p.* 28). This knowing, inherent to the profession of teaching, articulate with each other and is composed of knowledge, competencies, skills (or aptitudes) and attitudes of the professional teacher that make up a set of knowings that underlies the didactic actions in the school space (Tardif, 2002).

These perspectives lead to the understanding that teaching knowings is "[...] a plural knowing, formed by the amalgamation, more or less coherent, of knowing arising from professional training and disciplinary, curricular and experiential knowing" (Tardif, 2014, p. 36) and these go far beyond the knowledge that was taught in graduation (initial education), as they represent a typology of ideas articulated with each other through theory and practice.

In this context, in what follows, we present the characterisations of knowings and knowledge necessary for the teacher's teaching mobilisation. We first dialogue about them from the perspective of Tardif (2014) and Gauthier *et al.* (2013). Then, we approach the knowledge of the specialist or subject teacher with a view to the mathematics teacher related to Schulman's ideas (1986, 1987), intending to understand the elaboration of Ball, Thames, and Phelps' (2008) and Hill, Ball, and Schilling's (2008) criteria. Finally, we address the so-called didactic-mathematical knowledge of the mathematics teacher as presented in Godino, Batanero, and Font (2008), Godino (2009), Pino-Fan and Godino (2015), Godino *et al.* (2013), and Godino *et al.* (2017).

Teaching knowings from the perspective of Tardif and Gauthier

For Tardif (2002, 2014), teaching practice integrates different interrelated types of knowings and represents the combination of knowings arising from the professional, academic, curriculum, and experiential education. The author points out that teaching knowings is plural, heterogeneous, temporal, personalised and situated, carrying, in its constitution, marks that place it in an ethical and emotional perspective

Regarding the knowings of professional education (from the education sciences and the pedagogical ideology) we highlight that these refer to the "[...] set of knowings transmitted by teacher education institutions (normal schools or colleges of educational sciences" (Tardif, 2014, p. 36), which are legitimated in scientific theories.

About disciplinary knowings, Tardif (2014, p. 38) points out that it "[...] is also integrated into teaching practice through the education (initial and continuous) of teachers in the various subjects offered by the university [...]" , being recognised and identified as belonging to the different fields of knowledge, since it is part of the different subjects that aggregate the course plans within the scope of universities.

Curriculum knowings "[...] corresponds to the discourses, objectives, contents, methods from which the school institution categorises and presents the social knowings defined and selected by it as models [...]" (Tardif, 2014, p. 38), referring to a knowing that mentions the complexity of what is proposed in the formally established curriculum.

Finally, according to the author, experiential knowings "[…] springs from experience and is validated by it. They are incorporated into individual and collective experience in the form of *habitus* and skills, knowing-how-to-do and knowing-how-to-be" (Tardif, 2014, p. 39). It is practical, specific knowing, based on daily activities and reflections produced by the teacher.

As much as Tardif (2014) specifies that teaching knowings comes from professional education, disciplinary, curriculum, and experiential knowings, there is a specific knowing that results from combining all those types of knowings and is legitimated in daily teaching, which derives from different sources, articulated and mobilised by teachers according to the need for their teaching action.

Following the discussions that refer to teaching knowings, Gauthier *et al.*'s (2013) ideas stand out, identifying and characterising knowings that teachers should master, highlighting the disciplinary, curriculum, educational science, pedagogical, and experiential knowings, and knowings of the pedagogical action.

Disciplinary knowings "[...] refer to the knowings produced by researchers and scientists in the various scientific subjects [...]" (Gauthier *et al.*, 2013, p. 29), which are the curriculum components that make up undergraduate courses within universities, colleges, and others. Thus, it is pertinent for the teacher to know in depth the contents of the subject to be taught, because, "[...] research has been showing, more and more, that the type of knowledge that the teachers have of the subject influences their teaching and student learning (Gauthier *et al.*, 2013, p. 30) because one cannot teach something one does not master.

To Gauthier *et al.* (2013), disciplinary knowings undergo transformations by numerous agents (such as, for example, state secretariat employees, subject specialists, book authors and others) to become a teaching programme, i.e., to be qualified as a set of knowledge that makes up the curriculum knowings and, thus, "The teachers must know the programme, which constitutes another knowing of their reservoir of knowledge. It is, in fact, the programme that serves as a guide for planning, for evaluating" (Gauthier *et al.*, 2013, p. 31).

The knowings of educational sciences deals with the structuring of the school system, i.e., it refers to the basic, hierarchical, and administrative school organisation, political pedagogical project, school board, internal regulations, union, workload, course project, complementary activity, and notions of child development, among others. They are knowledge implemented in the school environment and allude to "[...] a set of knowings about the school unknown to most ordinary citizens and members of other professions" (Gauthier *et al.*, 2013, p. 31).

As for knowings that reference the teacher's habit and common sense, they are closely related to the private exercise of practice and the teacher's personal experience and cover aspects of experiential knowings, considering that "What limits experiential knowings is exactly the fact that it is made up of assumptions and arguments that are not verified through scientific methods" (Gauthier *et al.*, 2013, p. 29) and, to be recognised by researchers as specific knowledge of the teaching profession, they must be determined and validated by techniques, methods, and scientific research, and be published and disseminated for knowledge and experimentation by the teaching community.

Along the same lines of the existence of knowings arising from the teacher's experience and which is materialised in school, the knowings of the pedagogical tradition is pointed out. From this perspective, Gauthier *et al.* (2013) considers that the tradition of simultaneous teaching is installed in the making of the school when "The master stops teaching classes in the singular [...] he starts to practice much more simultaneous teaching, addressing all students at the same time […]" (p. 32), which can be exemplified by the organisation of the classroom with chairs arranged in rows to ensure silence and little interaction between students during classes.

Thus, the knowings of pedagogical action refers to the "[...] experiential knowings of teachers from the moment it becomes public and is tested by research carried out in the classroom" (Gauthier *et al.*, 2013, p. 33) and, therefore, constitutes knowings produced through the teacher's experimentation in his/her classroom when it becomes systematised, legitimised, and disseminated through scientific research.

The notes and reflections presented sought to highlight the teaching knowings arising from the thinking of Tardif (2002, 2014) and Gauthier *et al.* (2013), who, in a broad sense, discuss the knowledge inherent in the teacher's job. Tardif (2002, p. 54) highlights the view that the knowings of experience "emerges as a vital core of teaching knowings, from which teachers try to transform their interiority relations into their own practice", pointing out that the knowings from experience is not like the others, but formed by the others mediated by practice and experience (Tardif, 2002). Gauthier *et al.* (2013) defend the idea of the existence of specific knowings to the professional teacher, which is the knowings of the pedagogical action that arises from the interaction between the other types of knowings and the knowings that is the teachers' domain and that they use to answer the questions and requirements that emerge from teaching actions.

From Shulman's specialised knowledge to mathematics teacher knowledge

In what follows, we seek to produce discussions and reflections that focus on the knowledge mobilised by specialist teachers in specific subjects and, for that, the elaboration of these arguments was structured starting from the ideas of Schulman (1986, 1987), as per Ball, Thames, and Phelps (2008) and Hill, Ball, and Schilling (2008) who were based on Shulman's constructs to present a model of knowledge that the mathematics teacher must mobilise.

For the approach of disciplinary knowledge, Shulman (1987) published studies that emphasised the need to specify rhetoric of aptitudes to establish a professional status to the exercise of academic teaching. The author argues that there is a knowledge base that belongs to the specialist teacher and is represented by "[...] a codified and codifiable aggregate of knowledge, skill, understanding and technologies, of ethics and disposition, of collective responsibility – and also a means of representing and communicating it" (Shulman, 1987, p. 200).

Thus, it suggests that this so-called knowledge base for teaching should be part of the teacher's professional education, in the sense of mastering their ways of explaining ideas so that students come to understand them, of using teaching strategies and methodologies that enhance the students' learning possibilities, realising what should be learned and how to teach it. To Shulman

(1987), the knowledge that constitutes this framework refers to the qualities, skills, understandings, sensibilities, and postures that constitute a common subject in a competent specialist teacher.

In his studies in 1986, the author pointed out three categories of knowledge that are evidenced in the cognitive development of the subject teacher: content knowledge, pedagogical content knowledge, and curriculum knowledge. Later, in 1987, the author split those categories into seven: content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics and knowledge of educational contexts and, finally, knowledge of educational ends, purposes, and values.

Content knowledge refers to disciplinary knowledge specific to the teaching objects that will be addressed in the classroom. The general pedagogical knowledge incorporates the principles, methodologies, and strategies of teaching and classroom management, from which the teacher makes decisions on how to conduct their classroom. In addition to general knowledge, pedagogical content knowledge concerns linking the content with the different ways (strategies and methodologies) of teaching, whereas the knowledge of the learners and their characteristics concerns detecting students' difficulties to propose suitable ways of teaching and working with their difficulties.

Curriculum knowledge represents the programmes that specify the subjects (topics) by level of study, also including the instructional resources to be implemented in the classes and the knowledge of the educational contexts encompasses the functioning of the school, the classroom, the management, and financing of the educational systems, school council, characteristics of communities and their cultures. Finally, the author highlights the knowledge of the educational ends, purposes, and values and its historical and philosophical basis as important elements to be known by teachers.

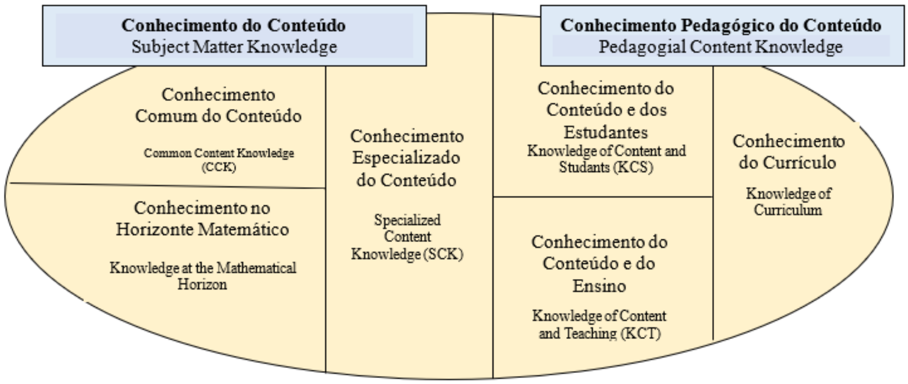
Among these categories, the author emphasises the relevance of the pedagogical content knowledge category, claiming that this is the specific domain of the specialist teacher since this involves articulating the content of the subject and the pedagogical knowings for the understanding of how to teach a specific object of knowledge successfully, highlighting that this knowledge is most likely the category that best distinguishes the understanding of a content specialist from that of a pedagogy professional (Schulman, 1987). This knowledge allows the specialist teacher (subject teacher) didactic-pedagogical reflections on the contents of a specific curriculum component.

Thus, Schulman (1987) points out that, despite the teaching profession being one of the oldest in the world, the systematic study of teaching is relatively new. Therefore, the defined teaching knowledge base is neither fixed nor definitive and many other types of knowledge can still be refined or aggregated because, with the advances in learning and scientific research "[...] we will begin to recognise new categories of performance and understanding that are characteristic of good teachers, and we will have to reconsider and redefine other fields" (Shulman, 1987, p. 213).

Considering Shulman's (1986, 1987) theoretical model, which indicates the knowledge covered by professional teachers of different disciplines, Ball, Thames, and Phelps (2008) and Hill, Ball, and Schilling (2008) resumed the discussion, as already highlighted, expecting to understand the teaching knowledge inherent to the mathematics teacher's job. To this end, the theoretical bases of Schulman's model (1986, 1987) were used to elaborate the model of mathematical knowledge for teaching (MKT), which refers to the mathematical knowledge teachers use in their practices, considering teaching actions aimed at student learning. Figure 1 presents the model of mathematical knowledge for teaching.

Figure 1

Mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008, p. 377).



The model considers two initial categories, content knowledge and pedagogical content knowledge, which, in turn, are organised into subcategories.

Regarding content knowledge, it is the “knowledge of the subject” (Hill, Ball, & Schilling, 2008, p. 377), that the teacher has of mathematics as a scientific discipline and presents three subcategories: common content knowledge, specialised content, and mathematical horizon knowledge.

The common content knowledge refers to the mathematical knowledge that mathematics teachers use at the teaching level, being also present in the lives of individuals and the context of different professions, serving to solve problems of professional applicability and common resolutions of everyday situations (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008).

On the other hand, specialised content knowledge concerns the “[...] mathematical knowledge that allows teachers to engage in particular teaching tasks [...]” (Hill, Ball, & Schilling, 2008, p. 377). Therefore, it is the knowledge that extends beyond teaching, demanding complex and refined mathematical reasoning processes (Ball, Thames, & Phelps, 2008).

Finally, mathematical horizon knowledge covers the understanding of how the content topics are correlated throughout the curriculum, recognising the integration of the topic being addressed, with those that were placed in the previous stage and with those that will be approached in the next step (Ball, Thames, & Phelps, 2008). “Having this kind of knowledge of the mathematical horizon can help in decision making [...]” (Ball, Thames, & Phelps, 2008, p. 403) before, for example, the need to review previous knowledge or address the concepts that interconnect future learning.

Pedagogical content knowledge, on the other hand, is related to the mathematics curriculum in articulation with the strategies, methodologies, and resources linked to the different ways of teaching, to the students’ learning phases, to school evaluation tactics, and to learning difficulties, among others. It is a domain that integrates the demand for specific content knowledge in connection with teaching practices. It is also presented in three subcategories: pedagogical content knowledge, knowledge of the learners and the content, and knowledge of the content and teaching.

Knowledge of learners and content is associated with the teacher’s possibility to anticipate students’ possible misconceptions, interpret their mistaken thoughts, and predict their possible errors and difficulties in the face of a specific task or new content. It refers to the ability to arouse positive emotions in students, and to make connections between common sense knowledge with systematised knowledge and others (Ball, Thames, & Phelps, 2008).

In turn, the knowledge of content and teaching concerns the professional skills of sequential structuring of content to be taught, recognition of the pros and cons of complex representation approaches, including the skills of adapting mathematical issues to the realities of students' cognitive demands.

Finally, curriculum knowledge presents how the contents are interrelated and how they should be structured throughout the school year, respecting the curriculum proposed by the school (Ball, Thames, & Phelps, 2008). The authors point out that this last knowledge is not yet fully defined in relation to its position in the model subcategory, as there are doubts about whether this should be part of the knowledge of content and teaching or whether it is a subcategory.

Ball, Thames, and Phelps (2008) and Hill, Ball, and Schilling (2008) argue that it is important to reduce the distance between what and how to teach mathematics and consider that mathematical knowledge and knowledge about teaching resources, strategies, and methodologies enable establishing connections between the concept and its applications. Understanding the content from different perspectives amplifies establishing relationships with other mathematical content and other areas of knowledge.

Regarding the specificity of mathematical objects and the relationships established in the teaching of mathematical topics and regarding mathematics teachers' knowledge, we highlight aspects of the model called didactic-mathematical knowledge based on the contributions of the onto-semiotic approach (OSA).

Teacher's didactic-mathematical knowledge

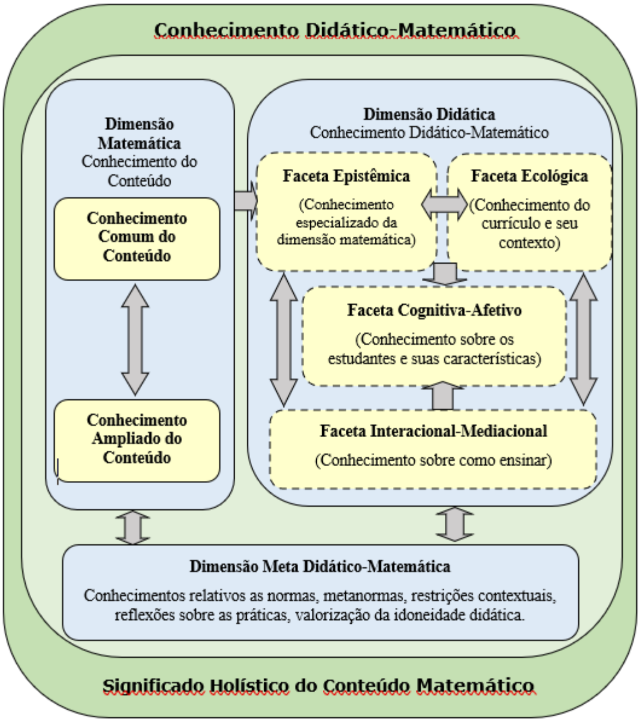
It was based on concerns such as, "How or with what criteria can knowledge be evaluated or measured? How can teachers be helped to acquire different knowledge? How do the different types of knowledge relate to each other?" (Pino-Fan & Godino, 2015, p. 93) that, within the scope of the OSA, construct that "[...] seeks to build theoretical tools to jointly analyse mathematical thinking, the mathematical objects that accompany it, the situations and factors that condition its development" (Kaiber, Lemos, & Pino-Fan, 2017, p. 535), the modelling of a "[...] system of categories to analyse the knowledge of the mathematics teacher referred to as didactic-mathematical knowledge” […]” (Pino-Fan & Godino, 2015, p. 95).

According to Pino-Fan and Godino (2015), this model began to be proposed in Godino (2009), based on the establishment of categories related to analysis tools proposed within the scope of the onto-semiotic approach, considering that this set of theoretical tools “[…] provides a system of categories and subcategories of knowledge that the teacher must know, understand, know how to apply and evaluate” (Pino-Fan & Godino, 2015, p. 96). The authors point out that in the guidelines presented for the creation of items to evaluate and analyse each of the dimensions and categories of the DMK “[…] a restructuring of the MKT model was implicitly introduced […] (Pino-Fan & Godino, 2015, p. 96).

Thus, considering that the knowledge to be mastered by mathematics teachers is plural and involves issues that are typical of teaching and learning mathematics, the model of didactic-mathematical knowledge (DMK) interprets and characterises the teacher’s knowledge from three dimensions: mathematical dimension, didactic dimension, and meta didactic-mathematical dimension. Figure 2 presents the DMK model, considering the dimensions mentioned, what they refer to, and how they relate.

Figure 2

OSA DMK dimensions and components. (Pino-Fan & Godino, 2015).



The mathematical dimension is related to the knowledge of the mathematical content "[...] that allows the teacher to solve mathematical problems and tasks" (Pino-Fan & Godino, 2015, p. 98) and, therefore, includes two categories of knowledge: common content knowledge and expanded content knowledge.

Common content knowledge concerns the contents proposed in official curriculum documents, in the school curriculum, in textbooks, constituting knowledge about a specific mathematical object and which is "[...] considered sufficient to solve problems or tasks [...]" (Pino-Fan & Godino, 2015, p. 97). On the other hand, expanded content knowledge (mathematical horizon knowledge) provides the teacher with the necessary mathematical bases to "[...] proposes new mathematical challenges in class, link the object being studied with other mathematical notions and forward the students to the study of notions subsequent to what is at the centre of the study" (Pino-Fan & Godino, 2015, p. 97).

About the didactic dimension, the authors point out that in addition to content knowledge, professional teachers must instruct themselves with specialised knowings on the many factors that influence planning actions to propose appropriate teaching strategies to enhance student learning and implement suitable resources and environments for student motivation. This dimension refers to didactic-mathematical knowledge, which, in the DMK model, is presented under the denomination of –epistemic, ecological, cognitive-affective, and interactional-mediational– facets.

The epistemic facet (specialised knowledge of the mathematical dimension) is the teacher's ability to present different strategies to solve the same task; display the different representations of the same learning object; understand and present the links of the object of the study presented here with those addressed before and with those that will be addressed later, either in the same or future teaching stages; move knowings to expose the various justifications and arguments in the face of doubt or the explanation of the solution of the same mathematical activity; identify the knowings at stake and the different resolution strategies when faced with a mathematical task (Pino-Fan & Godino, 2015). Therefore, "[...] it is the didactic-mathematical knowledge about the content itself, i.e., the particular way in which the mathematics teacher understands and knows mathematics" (Godino *et al.*, 2017, p. 96).

Regarding the cognitive-affective facet (knowledge of learners and their characteristics), it is '[…] knowledge of how students learn, reason, and understand mathematics and how they progress in their learning" (Godino *et al.*, 2017, p. 97) which is combined with knowledge "[...] students' affectivity, emotions, attitudes, and beliefs about mathematical objects and the study process" (Godino *et al.*, 2017, p. 97), resulting in a double-entry cognitive-emotional category, which gives teachers the necessary knowledge to reflect and assess students as people who think, who have their conceptions about the object of knowledge and who also have doubts and misconceptions on understanding the knowings taught.

The interactional-mediational facet (knowledge about how to teach) refers to the interactions between the teacher and the students, the students and the teacher and the didactic material, which also concerns to the organisation of activities in the perspective of identifying and resolving difficulties of students before[2] potential semiotic conflicts (Godino, Batanero, & Font, 2008), combining the management of technological, material, and temporal resources from the perspective of improving student learning (Godino *et al.*, 2017). Therefore, these facets integrate, develop, and enrich the notion of content knowledge mediated by resources and means that manage the teaching processes and aim to expand the students' learning possibilities.

Finally, the ecological facet (knowledge of the curriculum and its context) involves adjustments related to the curriculum factors of the course project, the relations of the object of study with other contents and with other subjects, the curriculum guidelines, the educational contexts, the conditions of the environment, purposes, and values of education related to mathematics instruction (Godino, Batanero, & Font, 2008).

The notes presented so far refer to the mathematical and didactic dimensions, with emphasis on the meta-didactic-mathematics dimension. This dimension concerns the knowledge that the teacher must have regarding the suitable theoretical foundations to reflect on the potential for improvement of the pedagogical practice, of the analysis of students' learning, aiming to improve the different ways of teaching and of valuing didactic suitability, because, "[...] in addition to the mathematical content, the teacher must know the various factors that influence the planning and implementation of the teaching of such mathematical content" (Pino-Fan & Godino, 2015, p. 98) to carry out teaching activities accordingly.

Godino *et al.* (2017) consider that "All these facets are part of a mathematics teacher's specialised knowledge to the extent that such processes bring into play some mathematical content, whether common or extended" (P. 97). In this sense, it is pertinent to take an investigative look at a mathematics teachers' training that provides for the constitution of teaching proposals from

[2] A semiotic conflict is any disparity or disagreement between the meanings attributed to an expression by two subjects (people or institutions). If the disparity is produced between institutional meanings, we speak of semiotic conflicts of the epistemic type, while if the disparity is produced between practices that form the personal meaning of the same subject, we designate them as semiotic conflicts of the cognitive type. When disparity occurs between the discursive and operative practices of two different subjects in communicative interaction (for example, student-student or student-teacher), we will speak of interactional (semiotic) conflicts (Godino, Batanero, & Font, p. 23, 2008).

the perspective of the constructs proposed by the onto-semiotic approach, from the perspective of the DMK, which is now presented.

ABOUT THE RESEARCH: METHODOLOGICAL ASPECTS

Based on a qualitative approach, this research was carried out along the lines of action research, conceived as a set of actions put into practice to enhance the teaching processes and expand the expectations of students' learning advances (Prodanov & Freitas, 2013). It aimed, as already mentioned, to investigate the development of knowledge and procedures related to geometry teaching with pre-service and in-service mathematics teachers. To this end, we planned and implemented a training course with 54 mathematics pre-service and in-service teachers from the Brazilian city of Paulo Afonso, Bahia, of which six work in the public school system and took over the supervision of the 48 pre-service teachers enrolled in teaching practice of the mathematics degree course at the State University of Bahia.

With the researchers' participation, the pre-service and in-service teachers constituted the representative members, engaged in a participatory and collaborative way in the proposal to mobilise knowledge from the perspective of initially structuring and then applying 24 sequences of activities (SA) in geometry, with teaching proposals for students in the final years of elementary school, intending to enhance their learning. The formative and investigative process was produced from May to October 2021, considering an interaction carried out via the *Teams* platform within the scope of so-called remote teaching due to the Covid-19 pandemic.

Thus, using the didactic-mathematics suitability analysis guide (DMSAG) (Table 1), which "[...] is, in fact, a family of instruments that summarise, in each case, the didactic-mathematical principles [...]" (Godino *et al*., 2013, p. 70), we collected some indicators for qualitative analysis of specific instructional processes, which make up the didactic suitability of the OSA. The adequacy criteria that make up the didactic suitability of a mathematical instruction process refer to the logical articulation of an intended or programmed educational practice that can be analysed from different degrees of adequacy (high, medium, low), "[...] However, this suitability must be interpreted as relative to unstable temporal and contextual circumstances, which requires an attitude of reflection and investigation on the part of the teacher [...]" (Godino, Batanero, & Font, 2008, p. 24).

Table 1

Facets of the didactic-mathematical suitability analysis guide - DMSAG (Adapted from Godino *et al*., 2013)

Epistemic Facet (AS structuring analysis)	**Other Facets** (AS application analysis)
Mathematical knowledge: The problem situations propose contextualisation, adaptation of the language to students' cognitive level and the different modes of representation of geometric entities. **Cognitive-affective knowledge:** The didactic paths and activities suggest awakening interest, articulating previous knowledge with new ones, and exploring contents from the simplest to the most complex topics. **Interactional-mediational knowledge:** Interactions between peers and exploration of various technological resources in implementing activities are related. **Ecological knowledge:** Adaptations are linked to the skills of the BNCC and the school curriculum, suggesting interdisciplinary connections.	**Cognitive-Affective Facet**: It refers to the ability to adjust the explanations of new content in connection with prior knowledge and management of difficulties, doubts, errors, and misunderstandings, proposing the development of interactions, self-esteem, and class participation. **Interactional-Mediational Facet**: It is related to the promotion of different modes of interaction in the classroom, identifies conflicts of meanings and learning difficulties, and concerns the use of manipulative and technological resources in the teaching process, developing the management competence of the class time. **Ecological Facet**: It refers to adaptations to the school curriculum, BNCC skills, didactic innovation, and interdisciplinary connections.

Thus, based on observations and experiences witnessed, the presence (albeit partial) of the indicators indicated in the guide was identified, considering the analysis of 24 SA produced. However, whenever necessary, specificities that were considered relevant and contributory to the theoretical reflections and dialogues that led to the topic of analyses and results were addressed.

It is also noteworthy that the components and indicators of the epistemic facet, function here as an instrument for the analysis of the structural aspects of the AS, and this protocol unfolds into four components (mathematical, cognitive-affective, mediational-interactional and ecological knowledge components) that evaluate the DMK mobilised by teachers when structuring the 24 AS. The other protocols, composed of the cognitive-affective, interactional-mediational, and ecological facets served as guides for the analyses of the knowledge mobilised in view of the application of activities to students.

The project was sent via Plataforma Brasil to the Ethics Committee for Research on Human Beings of the university where the research was conducted. It has been approved under substantiated opinion n. 4.442.096/2020 and CAAE 36285720.0.0000.0057.

RESULTS AND ANALYSIS

Here, results and analyses produced within the scope of the investigation are presented, which constitute part of the investigation. As already explained, the analysis was produced considering evidence of the didactic-mathematical knowledge mobilised by teachers in training from the production of 24 Sequences of Activities (SA), from the didactic-mathematics suitability analysis guide - DMSAG (Godino *et al.*, 2013). The results presented constitute, therefore, reflections on a synthesis of the analyses produced.

Epistemic facet analysis: SA structuring

The dialogues that follow conduct analyses that refer to the knowledge of teachers in training regarding the epistemic facet of the analysis guide (mathematical, cognitive-affective, mediational-interactional, and ecological knowledge), which encompasses remarks and reviews based on the indicators that refer to the teachers' common and expanded knowings, related to a specific topic of content, which, in this case, are particularly contemplated in each of the SA produced. Therefore, the content of the knowledge put into play (all referring to geometric knowledge) is not taken into account.

Thus, within the scope of the epistemic facet, the mathematical knowledge component was considered with medium adequacy because, in the face of the problem-situation indicator, although we observed that the questions were proposed by representations of "things" and situations of the physical world and that the identification of the characteristics of the mathematical object has been strengthened based on different forms of representation, these are not correctly configured as problem-situations. Problems with low levels of contextualisation were presented in exercise formats and, although relevant, should not be the only situations since "The problems cannot be excessively specific/isolated; instead, they should allow the articulation of the different mathematical competencies [...]" (Godino & Batanero, 2009, p. 6).

In this sense, we agree with Godino and Batanero (2009) when they state that "One of the main tasks of the mathematics teacher is to select and adapt problem situations that promote the contextualisation of mathematical content, its application, and exercise [...]" (p. 6) because this didactic action requires the teacher to develop the ability to analyse the connections of the teaching object with other areas of knowledge and with real situations, in order to link mathematical and didactic skills, considering aspects of the focus (OSA).

Regarding the analysis of the cognitive-affective knowledge component, we noted substantial evidence of articulation of previous knowledge with current knowledge to expand learning and adapt planned activities and resources to students' cognitive levels and teaching stages. Such adaptations were conducted by didactic paths that valued the exploration of contents from the most uncomplicated knowings to achieve student learning before more complex knowledge. We observed that teachers invested in the insertion of digital learning objects suited to the demands of the online classes, with evidence of tasks with game performances, which sought greater interactivity and awakening of interest. Thus, regarding cognitive-affective knowledge, we identified high suitability.

Regarding the interactional-mediational knowledge component, this, too, was qualified as highly suitable, as we observed that the remote classes were planned with apps such as *Google Slides*, *PowerPoint*, and *Canva*, where it was possible to perceive the constant search for better organising the objects of study, maximally exploring different forms of representations, aiming at a work focused on elements of visualisation and movement, making tasks (and evaluations) available with attractive performances in the eyes of the learning subjects.

For that, the activities were made available through *Google Forms* and *Liveworksheets* [3], digital platforms that transform traditional activities into interactive exercises with direct self-correction provided by the site. Other tasks were turned into games, through the platforms made available for this purpose, such as *Kahoot*[4] and *Wordwall*[5]. Others were made available through *software*/platforms such as *Geoboard*[6], *Geogebra*[7] and *Poly*[8]. Thus, they

[3] *Liveworksheets*: https://www.liveworksheets.com/.

[4] *Kahoot!*: https://create.kahoot.it/auth/register.

[5] *Wordwall*: https://wordwall.net/pt.

[6] *Geoboard*: https://apps.mathlearningcenter.org/geoboard/.

[7] *Geogebra*: https://www.geogebra.org/?lang=pt

[8] *Poly:* http://www.peda.com/poly/.

exercised the development of digital competence, considering that professional teachers "[...] need to establish mechanisms and strategies to introduce these tools in educational processes and study the consequences of this introduction" (Larios *et al*., 2012, p. 25).

Regarding the ecological knowledge component, this was also configured with high didactic suitability. The contents developed in the SA were proposed and presented with activities and didactic paths designed with adaptations to the skills of the National Common Core Curriculum Base (BNCC) (Brasil, 2018) and to the schools' study plans, which allowed to perceive the intentionality of the prospective teachers in establishing connections between mathematical content and proposals in official documents that suggest relating school knowledge and mathematical instruction processes to students' learning demands.

The results, analyses, and arguments presented point to the understanding that the epistemic facet, in a broad sense, is evaluated with high didactic suitability, taking into account that only one of the indicators (problem situations) of the mathematical knowledge component showed weakness and was analysed with average suitability.

Analyses of the other facets: application of the SA

The following analyses refer to the cognitive-affective, interactional-mediational, and ecological facets and concern the applications of the SA by the pre-service teachers, accompanied by the teachers in charge of the classes and by the researcher, who carried out observations while participating in the online classes.

The systematic follow-up of the remote meetings allowed us to perceive evidence of a high degree of suitability of the cognitive-affective knowledge. We noted that aspects of adapting classes to the previous knowledge necessary for developing new content and managing difficulties students had during classes were adequately conducted. The same happened with handling students' doubts, errors, and misconceptions during class participation. At those times, the teachers used different mediating elements (specific videos produced and a digital table) to meet the requests and needs of the students, articulating prior knowledge with new approaches. It was also possible to perceive a constant movement in the development of curriculum adaptations, strategies, and teaching methodologies, aiming to awaken the student's interest in participating in the class, although it did not always occur as much as we would like.

Planned didactic actions showed an interest in raising positive feelings in the students, such as developing workshops for the construction of geometric solids with manipulative materials (toothpicks, candies, cardboard). On these occasions, we could perceive that the students actively participated in the event and even answered questions about the characteristics and terminology of the geometric objects. We consider that these situations amplified feelings of self-esteem and motivation for learning mathematics, and we perceived significant and better-quality students' participation.

About what was pointed out by the analysis guide as suitable for the interactional-mediational facet, particularly regarding the management of interactions in the classroom, we observed that most teachers showed high suitability concerning the communication of mathematical ideas, explaining clearly, emphasising key concepts and following with paused speeches and logical organisation of thoughts. This coherence was always guided by the didactic materials used for the classes that included open questions to be solved, considering group work.

Regarding the management of material resources, it was possible to identify that the teachers were concerned with using diversified teaching resources and strategies mediated by digital and non-digital teaching resources, including manipulative ones. They used different learning objects and proposed activities in digital game formats. Among the digital resources used, we highlight *Poly* and *GeoGebra* dynamic geometry *software*, study materials prepared with *PowerPoint* and *Canvas*, and learning objects organised from digital platforms such as *Poly, Geoboard*, *Wordwall* and *Kahoot!* to expand the possibilities of students' inclusion in the classroom dynamics.

The evidence and arguments presented make it possible to establish that the pre-service teachers mobilised the necessary knowledge "[...] to anticipate, implement, and evaluate sequences of interactions between the agents involved in the teaching and learning processes, aiming at the fixation and negotiation of students' meanings (learning)" (Pino-Fan & Godino, 2015, p. 101), which allows considering high didactic suitability with regard to what is established in the interactional-mediational facet.

Finally, regarding the ecological facet, we could observe high suitability as the pre-service teachers put into practice the necessary adaptations of what was established in the study plans with the skills presented in the

National Common Core Curriculum Base (Brasil, 2018), showing " Competence in the search, selection, and adaptation of good practices that involve the use of the real context and interdisciplinarity" (Godino *et al.*, 2013, p. 58). Still, expecting to motivate students and promote their progress, the teachers also made curricular adaptations regarding the didactic trajectories, which were traced following a logic of thoughts that started from the survey of previous knowledge, followed by the articulation with the new knowings, and ending with their synthesis.

And about the openness to didactic innovation, teachers showed high didactic suitability and a "Favourable but reflective attitude towards research-based innovation" (Godino *et al.*, 2013, p. 58), since the activities were proposed based on the structuring of an AS, considering research in productions focused on the teaching of geometry, and whose tasks were implemented considering different teaching resources, as already mentioned.

Furthermore, high suitability was present, based on the mobilisation of knowledge on issues involving cultural values and citizenship in the context of approaches in the classroom, when the pre-service teachers used small texts and videos found on *Youtube* and previously selected, which were focused on bringing the indicated contexts to the students, with a view to producing a space for discussion, interaction, production of knowledge, and greater interaction. Such situations made it possible to explore intra[9] and interdisciplinary[10], from the perspective pointed out by Larios *et al.* (2012) when he emphasises that the teacher must "[...] use the educational and sociocultural value of mathematics and its historical evolution in the construction of a mathematics activity, and relate it to the different teaching and learning proposals" (p. 34).

Finally, we consider that the proposed training process, the studies, discussions, and reflections that culminated in the productions and applications of the AS, enabled the pre-service and in-service teachers involved to use and

[9] Intradisciplinary "It is a relationship between the objects of internal knowledge of the curriculum component itself, i.e., how contemporary cross-cutting themes permeate within the skills of the different thematic units presented" (Colares & Santa Cruz, 2021, p. 2).

[10] Interdisciplinary "It is an integrated approach to contemporary cross-cutting themes common between different curriculum components. It implies a dialogue between the fields of knowings, in which each component welcomes the contributions of the others, i.e., there is an interaction between them" (Colares & Santa Cruz, 2021, p. 2).

deepen skills to plan and apply teaching activities in a context that allowed to "[...] integrate theories, methodologies, and curriculum in the planning of teaching processes and recognise the implications in their practice considering the institutional contexts" (Larios *et al*., 2012, p. 32), within the scope of what was identified as a legitimate process of development of didactic-mathematical knowledge, as presented in Pino-Fan and Godino (2015).

CONCLUSIONS

The theoretical reflections, which made it possible to trace the trajectory on the proposal of theoretical models that support teachers' knowings and knowledge, particularly of mathematics teachers, allowed us to perceive that the specialised mathematical and didactic knowledge already in the domain of the group, and others built in the process, led to the structuring and application of activities, paths, strategies, and teaching methodologies and the insertion of materials adapted to students' cognitive levels, considering that the experience was not a solid element with the pre-service teachers.

The analyses produced based on the OSA constructs referring to the teacher's didactic-mathematical knowledge, based on the didactic-mathematics suitability analysis guide (Godino *et al*., 2013) pointed to a high didactic suitability regarding the productions and applications of the SA. Not all indicators have obtained a mention of high suitability in the analysis, but, considering a global view, we can say that the productions carried out throughout the training process led to it. We think that the pre-service and in-service teachers must seek to develop and qualify the specific knowledge of the profession to be capable of producing didactic-pedagogical actions based on theoretical knowings, articulated with its citizen commitment to teaching.

It is pertinent to approach and suggest that projects or training programmes for mathematics teachers be organised in a way that awakens in the participant a sense of belonging; that the participants are heard in their suggestions, valued in their difficulties, and assisted in their needs. Furthermore, the participants should be called to participate in the development of the training activities actively. Therefore, this action creates an environment of collaboration and co-participation, given the perspectives of qualifications (training) and the mobilisation of didactic-mathematical knowledge from their realities (research). All this needs theoretical support, and, in this sense, the constructs of the DMK-model onto-semiotic approach are pointed out as promising. However, as Pino-Fan and Godino (2015) state, the model must not be unique and closed; it must be open to new perspectives and understandings.

AUTHORSHIP CONTRIBUTION STATEMENT

SFMP and CTK conceived the idea presented; adapted the methodology for this context; implemented data collection guides; collected and analysed the data; actively participated in the discussion of the results and, finally, reviewed and approved the final version of the work.

DATA AVAILABILITY STATEMENT

The data collected for this research will be made available through contact with the author SFMP, with adequate justification.

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