

Mathematical Reasoning Required when Students Seek the Original Graph from a Derivative Graph

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Received for publication on 11 Jun. 2020. Accepted after review on 22 Oct. 2020

Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: Finding the original graph when given the derivative graph is not a trivial task for students, even though they can find the derivative graph when given the original graph. **Objective:** In the context of qualitative research, this paper presents and analyses the mathematical reasoning that comes to light when the students seek the original graph from a derivative graph. **Design:** The research is assigned as a qualitative study, where the analyses of cases aim to extend understanding with respect to some phenomena or theory. **Setting and participants:** The study was conducted with 86 students from a State University in East Java. We conducted clinical interviews, and present data highlighting the reasoning participants used when solving tasks. **Data collection and analysis:** Task-based interviews were used to collect data, and data analysis was used to analyse interpretations of the graphs that emerged as mathematical reasoning models. **Results:** From our data analysis, we found that three mathematical reasonings were rooted in students' awareness of problem-situations on graphs we provided, consisting of direct reasoning, reversible reasoning, and combined direct-reversible reasoning. **Conclusions:** We suggest that there are different mathematical reasonings in the construction of the original graph, due to the mental activity in which students use the relation between a function and its derivative. We suggest that future projects continue this inquiry with rigorous single-subject experiments with students.

Keywords: Mathematical Reasoning; Calculus; Derivative Graph; Direct Reasoning; Reversible Reasoning

Raciocínio Matemático Necessário para Estudantes que buscam o Gráfico Original a Partir de um Gráfico Derivado

RESUMO

Contexto: Encontrar o gráfico original ao partir do gráfico derivado não é uma tarefa trivial para os alunos, mesmo que eles sejam capazes de encontrar o gráfico derivado dado o gráfico

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original. **Objetivos:** No contexto da pesquisa qualitativa, este artigo apresenta e analisa o raciocínio matemático que surge quando os alunos buscam o gráfico original a partir de um gráfico derivado. **Delineamento:** a pesquisa é caracterizada como um estudo qualitativo, onde as análises de casos visam ampliar o entendimento a respeito de alguns fenômenos ou teoria. **Cenário e participantes:** O estudo foi realizado com 86 estudantes da Universidade Estadual de Java Oriental. Realizamos entrevistas clínicas e apresentamos dados destacando o raciocínio utilizado pelos participantes na resolução de tarefas. **Coleta e análise de dados:** Entrevistas baseadas em tarefas foram usadas para coletar dados e a análise de dados foi usada para analisar as interpretações dos gráficos surgidos como modelos de raciocínio matemático. **Resultados:** Em nossa análise de dados, descobrimos que três raciocínios matemáticos estavam enraizados na compreensão dos alunos sobre as situações problema dos gráficos que fornecemos, consistindo em raciocínio direto, raciocínio reversível e uma combinação de raciocínio direto e reversível. **Conclusões:** Sugerimos que há um raciocínio matemático diferente na construção do gráfico original devido à ação mental em que os alunos usam a relação entre uma função e sua derivada. Sugerimos que projetos futuros continuem esta investigação com rigorosos experimentos com alunos sobre um único tópico.

Palavras-chave: Raciocínio matemático; Cálculo; Gráfico Derivativo; Raciocínio Direto; Raciocínio Reversível

INTRODUCTION

Derivative studies have had increased attention from researchers in recent years. Much attention has been given to the understanding of a derivative in a graph situation, such as sketching an original function with properties of a known derivative or vice versa (García et al., 2011; Hong & Thomas, 2015; Natsheh & Karsenty, 2014). Sanchez-Matamoros (2014) provides information on very important aspects about the learning of the derivative, such as: errors and difficulties in its understanding; relations between the rate of change and incremental quotient; systems of representation as tools for thinking about derivatives; the relations between derivative at a point and the derivative function; the development of the derivative schema; the application of the concept in other problems (for example, the development of the understanding of the chain rule). Most empirical research in mathematics education has summarised students' conceptions of graphs (Carlson & Thompson, 2017) and their difficulties (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Baker, Cooley, & Trigueros, 2000a; Berry & Nyman, 2003), while others have offered a comprehensive analysis of students' strategies (García et al., 2011; Haciomeroglu et al., 2010) and models for them to understand the graph of the function (David et al., 2018). However, reasoning from a derivative graph and analytical properties is not well-documented in the literature.

Sketching graph f based on the properties of its derivative has been the main focus of mathematics studies in some countries. Torner, Potari, and Zachariades (2014) discovered that some European countries (Greek, Cyprus, Italy) introduced the idea of sketching derivative and antiderivative graphs, but failed to study it in-depth (limit and continuity) so that the studies were more likely to focus on procedural aspects of sketching graphs. On the other hand, research by Natsheh and Karsenty (2014) pointed out that most of the Palestinian students were unable to sketch a graph based on its derivative properties due to their incomplete visual reasoning. However, Indonesian students are rarely faced

with the task to sketch graph f based on graph f' because learning is more focused on procedural aspects (e.g. algebra-symbolic thinking), despite the fact that calculus learning emphasised the development in interpreting graphs, applying properties of derivative with analytical and graphical conditions.

In prior studies, there seems to be an agreement that understanding a concept involves the ability to interpret concepts from various representations, including from verbal, numerical, and visual ones (Berry & Nyman, 2003; Ryberg, 2018; Zandieh & Knapp, 2006). Graphs are one of the essential visual representations that can link various topics in mathematics, for instance, representing relationships between quantities, searching for the behaviour of functions, seeking unknown values, and exploring geometric transformation (Moore et al., 2013). Even though students are algebraically proficient in applying a mathematical algorithm to find the derivative of a function, they possibly find it difficult to draw or interpret problems involving graphs. This is in line with the findings by Berry and Neyman (2003), who state that most students can find derivatives that produce zero, but they fail to understand the importance of the derivatives. In this case, the students cannot understand the meaning and relationship between functions and derivatives graphically.

Research has paid special attention to charts for calculus problems. Ubuz (2007) explains that there are cognitive processes related to graphics, namely construction, and interpretation. Both are important parts to understand the basic concepts of calculus, where interpretation refers to the mental action of students to read and interpret graphs, while construction refers to the mental action of producing something different, constructing graphs, or plotting points based on data. Sofronas et al. (2011) put graphical understanding as a relevant aspect for students in early-grade calculus classes to understand the concept of derivatives which involves the concepts of gradients, extremes, turning points, and concavity. However, some findings show that most students fail to construct central ideas of functions and derivatives graphically, for example, students fail to relate sign of first derivative and second derivative, continuity, and limit values to compile the original function. Although most students can describe that derivatives provide information about the slope of original function, they have trouble applying it to graphical problems.

In undergraduate mathematics, construction and interpretation of graphs are pivotal pieces of learning for reasoning the visual ideas of function and its derivative. However, some studies provided evidence that students often interpret and construct mathematical ideas with analytic and symbolic rather than visual representation as they solve a graphical task. Haciomeroglu et al. (2010) found that students who have a strong preference for symbolic and analytic thinking did not need a visual representation and vice versa. Many students can sketch a graph correctly from analytic, symbolic, and numerical representations, but they were unable to recognise their crucial ideas. Gray and Tall (1994) explain that the conditions are due to lack of proceptual encapsulation and reversible thinking when connecting the function and its derivatives.

However, there are several possible ways in which students may conceive the visual meaning of function and its derivative, such as monotonic, which is a word frequently

related to a visual representation. The monotonic function can be expressed as “let f be continuous on an interval I and differentiable at every interior point of I . Then, if $f'(x) > 0$ for all x interior to I , then f is increasing on I ”. This monotonic theorem is more frequently used to find where f is increasing if the function of f exists than by sketching the graph of f without verbal, numerical, or symbolical information from the graph of f' . On the other hand, seeking an original graph from a derivative graph is not a trivial task for students, since they need to anticipate, coordinate, and reverse relationships between the function and its derivative.

In answer to high failure rates nationwide in derivative courses, a great deal of research between the 1990s and the 2000s revealed that students find difficulties in understanding derivative concepts with a graphical representation. Such research has contributed to the perception of the students' general difficulties, regarding the concepts of derivative, in interpreting the sign of the first derivative, the cusp point, discontinuity, and the second derivative to determine the concavity. In short, the research has focused almost exclusively on the fact that reversing between function and its derivative is not simple.

Few studies have documented student's conceptions in graph of a function and its derivative (David et al., 2018; García et al., 2011; Natsheh & Karsenty, 2014; Sofronas et al., 2011), such as the derivative of a function at a point, the slope of the line tangent to the graph of the function at that point, tangent as the limit of a set of secant lines. These studies consistently report that procedures of graph sketching did not develop in students an understanding of the underlying ideas. Several studies indicate students' dilemma in handling the given meaning to the second derivative, removal of discontinuity, vertical tangent at a point, and cusp and inflection point. Haciomerouglu et al. (2010) showed that students' preference for differential calculus demonstrated dissimilar difficulties. For example, the analytical thinkers had problems with discontinuity and differentiability while drawing graph f , on the other hand, the visual thinkers had problems with interpreting how the vertical stretching of graph f changes graph f' . When students deal with those situations, they should reconstruct and reorganise prior knowledge, and consciously use the relationships between the behaviour of f and the sign of the first derivative to the behaviour of f' and the sign of the second derivative (García et al., 2011).

In summary, this research is relevant and necessary, considering the following: (1) mathematical reasoning and sketching a graph are a current theme in mathematics education research programmes, as reported in mathematics education literature; (2) mathematical reasoning and sketching a graph should be encouraged in the classroom according to the curriculum at the pre-university level of several countries (including Indonesia); (3) mathematical reasoning and sketching a graph involving the derivative graph allow us to explore the mathematical understanding of a student regarding the meaning of the derivative in graphical context; and (4) this research provides relevant information to understand the types of students' mathematical reasoning and can serve as a theoretical framework to study the mathematical reasoning in future research. So, this research has two main contributions to the field of mathematics education: first, it

provides a framework for studying students' mathematical reasoning in graphical context, as the types of mathematical reasoning that students can make in sketching a graph have not yet been identified. Second, it demonstrates the usefulness of thematic analysis to perform a qualitative analysis of the written and verbal productions of students that involve mathematical tasks. The literature review in this part assist in distinguishing specific aspects of the student's conceptions about graphs; however, current literature does not explain aspects of mathematical reasoning in sketching graphics. Thus, it is acceptable and informative to identify mathematical reasoning finding an original graph from a derivative graph. Through analysing student's reasoning of statements using a graph, we strive to answer the research questions: what is the mathematical reasoning needed to find the original graph from a derivative graph?

METHODOLOGY

In studying an individual's apprehension for various mathematical ideas, researchers are at a disadvantage, and we cannot see how he/she thinks. Rather, we only have identified what an individual says when he/she thinks aloud, writes, and gestures when involved in mathematical tasks. Therefore, we focus on the individual's expressed ideas or the spontaneous expression that states an idea. We can make inferences about his/her experiences and reversible reasoning with the notion. Thus, we conducted qualitative research to highlight a particular issue in-depth. The research is assigned as a qualitative study, where the analyses of cases aim to extend understanding concerning some phenomena or theory (Yin, 2014).

Task-based interviews were used to explore the variety of their answers, conceptions, and reversible reasoning. The set of these interviews and their concept constitute the core of qualitative research.

Participants

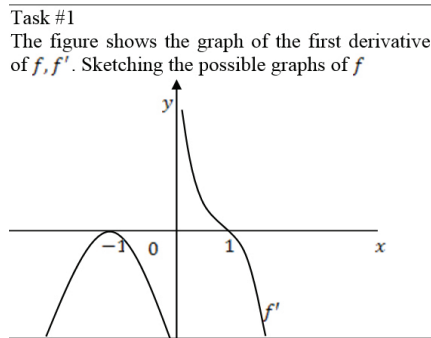
This study was organised in one section of the course "Research on school mathematics" at a State University in East Java during the regular semester of 2018. Out of 86 students registered, three selected in the research were between 19 and 21 years of age. The students were selected due to their easy access to the studies team in terms of location, program, and scheduling. Furthermore, we selected the participants who had completed an advanced calculus course and had prior knowledge in working with calculus in the graphics context. The participants were given pseudonyms S1, S2 and S3 to have their identities preserved. We asked them to answer the task as we present below and analysed what they thought when solving the situation.

Interview Tasks

The research team involved four researchers who assisted in specific roles. During each interview, one of the researchers performed as the interviewer, and the other served as a supervisor. All interviews were held in Bahasa, in a study hall at the university during the lunch break. The respondents' expressions were recorded and transcribed. Each interview was recorded with digital cameras, one focused on the verbal expression (think aloud) and one focused on the respondents' written work. These videos were digitally combined into one file for data analysis. Thus, we conducted a clinical interview designed to (1) gain a deeper understanding of students' solution process of all the proposed tasks, and (2) to investigate the type of mathematical reasoning that a student requires when sketching graphs.

During the interview, all participants were asked to verbalise their conceptions when solving the tasks. Thinking aloud is considered to give acceptable information without interfering with thinking or cognitive process (Ericsson, 2006). The interviewer asked participants to sketch the original graph from a derivative graph, as presented in Figure 1. The task comes from the book entitled "Calculus Ninth Edition," by Purcell et al. The objective of this task was to obtain information regarding students' perceptions of the interpretation of a derivative graph, and the properties of derivative to sketch an original graph. The derivative graph does not contain numerical or symbolical information; therefore, the root of information is only visual. We determine the general point which is specified with special points as $x = -1$, $x = 0$, and $x = 1$. To sketch the original graph, we expected that students worked in derivative rather than integration perspectives. Students possibly using the meaning of derivatives as the slope of the tangent line to the curve of function, making a relation between the sign of the first derivative with the monotonic of f , and establishing a relation between differentiability and continuity. In Task #1, students were given the graph of the function f' , which includes several sign changes; increasing and decreasing slopes, zeros, extreme values, and a discontinue point. The task asks students to sketch a graph for the function f , and its objective is to assess whether students could coordinate the processes related to the extreme values of f' with the processes associated with the extreme values and inflection points in f . Similarly, to coordinate the processes associated with the sign of f' with those related to the monotonic of f and finally, to coordinate the processes related to the increase of f' with those related to the convexity of f .

Figure 1
Examples of proposed problems



Data Analysis

We initially used inductive content analysis for the data. We identified emerging themes of the participants' conceptions, with particular attention to their mathematical reasoning when sketching the graph of f based on the graph of f' . This part containing the data collection and data analysis is rooted in a constructivist perspective. This perspective interprets that researchers can only model their interpretations of graphs based on expressions and visible behaviour. Our analysis produced hypothetical models of students mathematical reasoning when sketching the original graph grounded in observable behaviours, including their words and gestures.

The data analysis was composed of three phases. In the first phase, *preliminary analysis*, data analysis started from transcript think-aloud and interview. During each interview, we built initial assumptions based on verbal expressions and gestures. These initial presumptions were engaged to guide the interviewer's follow-up questions. After each interview, we met with the research team to discuss the resulting interview. In the second phase, called *Open Coding*, we interpreted the video data student by student to find patterns that explain their mathematical reasoning when seeking the original graph from a derivative graph. Through our analysis via open coding, student interpretations of the graphs emerged as mathematical reasoning models, for instance, by employing a symbolic approach to solve the problem, or by employing a visual approach to construct the original function.

We decompose the video by expanding codes to depict interesting and relevant aspects of the mathematical reasoning about seeking the original graph. We utilised these codes to analyse the other video to refine and develop the initial coding to explain mathematical reasoning. We elaborated the results of the interviews, applied the codes, and restructured the previous analysis. At the end of this open coding process, three significant codes appear that mark mathematical reasoning when seeking the original graph from derivative graphs, which consists of direct reasoning, reversible reasoning, and combined direct-reversible reasoning.

In the third phase, once these direct reasoning, reversible reasoning, and combined direct-reversible reasoning had been developed in *open coding*, we corrected this category using the *axial coding*. To correct our definitions, we compared characteristics of students thinking (for instances, *activating and anticipating*) to check for consistencies in their ideas of that category. Additionally, we compared diverse categories as a means of further ensuring our finding. Through this manner, we built a description of each mathematical reasoning when seeking the original graph from a derivative graph that emerged in our analysis.

RESULTS

In total, 74 of the 86 participants denoted consistent methods to determine the graph of f with symbolic processes. Twenty-seven students initially considered such questions difficult because of the lack of algebraic representation. Furthermore, fourteen students encounter difficulties when interpreting the graph of f' . Twenty-one students assume the graph of f' is a polynomial and they were trying to find the general formula for such a polynomial function using $y = a(x - b)(x - c)$, finding symbolically the antiderivative of f' , and sketching the graph of f .

Interestingly, twelve students used an interval approach with assuming the derivative graphs comprise two graphical representations (graph on the interval $(-\infty, 0)$ represent an even function or x^2 , x^4 , or x^{2n} or and graph on the interval $(0, +\infty)$ represent an odd function or x^3 , x^5 or x^{2n+1}). Presuming the problem situation depicts reverse quadratic and cubic graph or symbolically represent $f'(x) = -(x+1)^2$ and $f'(x) = -(x-1)^3$ until developing reverse operation to find the graph of f . However, 74 participants did not interpret graphically the tasks presented, such as the process of x moving through an interval.

Twelve other students, as shown in Table 2, denoted a consistent method to determine the graph of a function with reverse processes. They consider that finding the graph of f is a novel situation; therefore, there was mental reconstruction to reverse the problem-situation. Most of them were able to interpret location and value of the derivative graph, such as: a curve lies above X -axis on an interval $(0,1)$ and below x -axis on interval $(-\infty, 0)$ and $(1, \infty)$, graph f' on an interval $(-\infty, -1)$ is increasing, and the other interval is decreasing, graph f' on interval $(0,1)$ is concave up, and the other interval is concave down. Explicitly they justify the relationship, saying that “if f' is positive because f' is the slope of the tangent to the curve of f , then if the slope is positive, f is increasing.”

Table 2*Students Performance (# denotes the number of students who exhibited the conception)*

	#
Critical points as the first derivative is zero $f'(x) = 0$	6
The slope of the tangent line	3
Coordinating monotonic and concavity	2
Steepness and direction of a line	1
Removal continuity condition	11
Singular point and vertical asymptote	1

Six students could identify the stationary points at $x = -1$ and $x = 1$ as the first derivative is zero. Three students talked about the “the slope of the tangent line” to infer the whole behaviour of the graph of f . Two students were aware of the coordination of the relationships between the value of f' or the sign of the first derivative with the behaviour of f and the coordination of the increase/decrease of f' , the sign of the second derivative, and concavity of the function to sketch the original graph. The other student described the derivative graph, considering the steepness and direction of the lines.

11 of 12 students disregarded the discontinuity condition of graph f' at $x = 0$, for the lack of reversible thinking to coordinate the relationship between continuity and differentiability in a graphical context. S1 is one student who consistently exhibited those relationships as a singular point (sharp-turn) and vertical asymptote, resulting from the discontinuity of the derivative graph at $x = 0$ (Figure 2). On the other hand, S2 combined direct-reasoning and reversible-reasoning to solve a problem (Figure 3). S3 predicted the model of graph f' through an integration process and proved the model using derivatives properties (Figure 4).

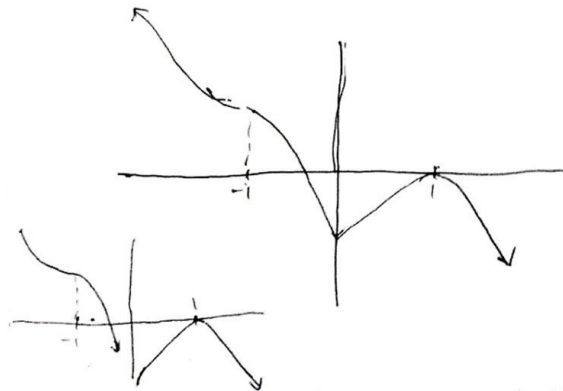
Figure 2*S1 answer to task*

Figure 3
S2 answer to task

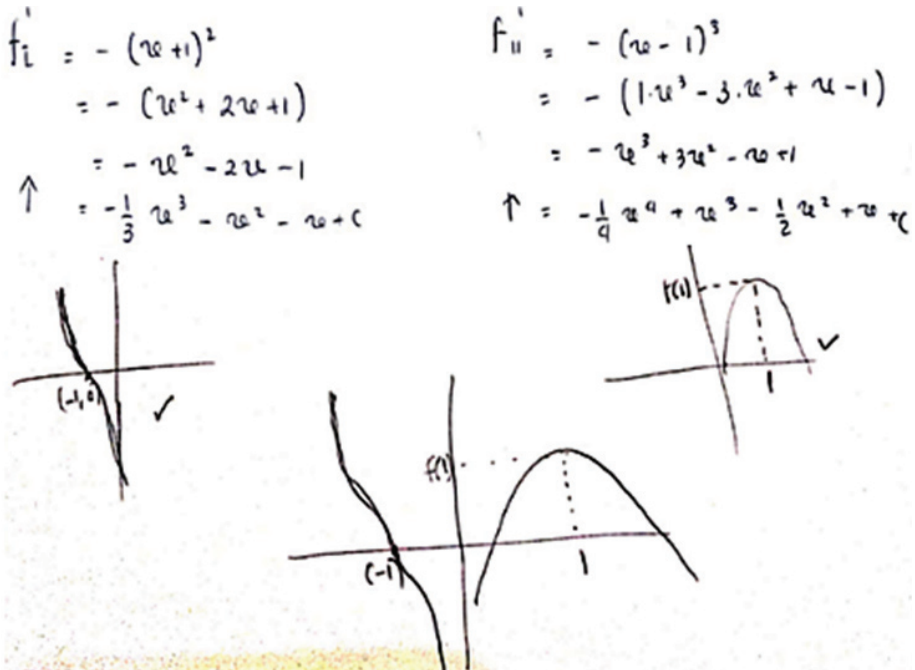
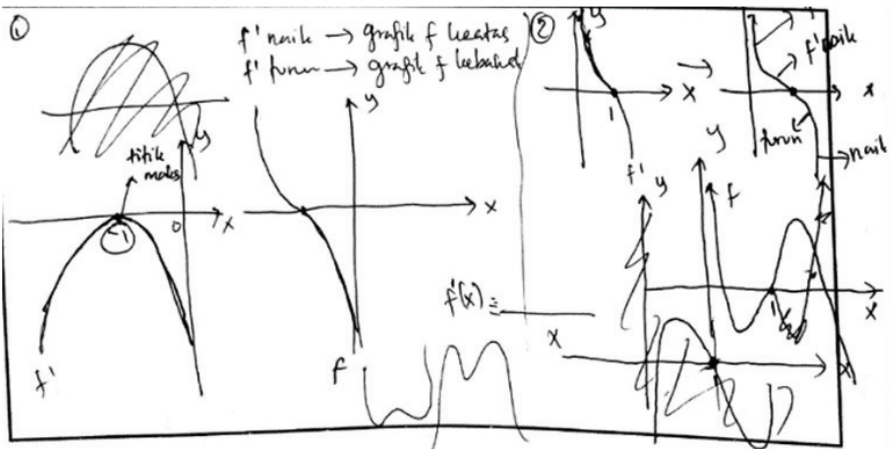


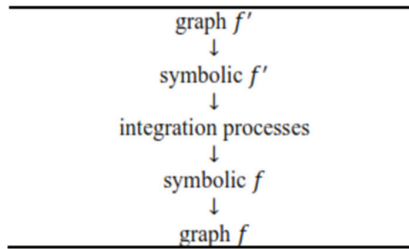
Figure 4
S3 answer to task



DISCUSSIONS

Based on our study, we conclude that the differences in the place of a student's focus when reasoning about a graph have significant implications on their understanding of related mathematical ideas. This section discusses findings related to the participants' mathematical reasoning when sketching graph f based on graph f' , that was through direct-reasoning and reversible-reasoning. The first category of direct reasoning was shown in the subjects' mental activity in the integration process of obtaining graph f . The subjects thought $f' \rightarrow f$ using symbolic representations. We can also say that when the subjects encountered a problem, they immediately realised the importance of finding the formula of graph f' function, of doing an integration process to discover the formula of function f , and sketched graph f , as in Figure 5

Figure 5
Direct-reasoning pathway



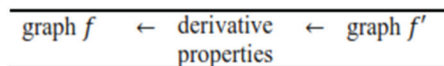
This category is corroborated with Hacıomeroglu's opinion (Hacıomeroglu et al., 2010), that that says that when solving problems dominated by analytic approaches, subjects' thinking process can be limited by the absence of support of a visual approach. In other words, an analytic approach results in subjects routinely translating problem situations into symbolic representations, so they tend to estimate the equation from derivative graphs. Subjects who possess this ability are highly dependent on the cognitive style that prioritises verbal-logical processes and ignores visual processes (Hacıomeroglu, 2016). Similarly, Hong, and Thomas's findings (2015) support that direct-reasoning is identical to symbolic process algebraic thinking, the type of thinking that is useful in manipulating a model, but tends to be detrimental when the problem given is dominated by verbal aspects. The presence of visual representations allows these types of thinkers to experience obstacles due to their inability to transform problem situations (Montenegro et al., 2018). Besides, this type of thinking is caused by the lack of proceptual encapsulation (Tall, 2008), where the thinkers cannot see differentiation and integration as an equal process.

The next category is found in the subjects who perceived graph f' as the result of graph f . Therefore, this type of thinking is categorised into reversible reasoning. Subjects

in this category considered finding graph f based on known graph f' by utilising its derivative properties, as in Figure 6.

Figure 6

Reversible-reasoning pathway



This is in line with Steffe and Olive’s (2009) opinion, that refers to reversible reasoning as an individual’s effort in constructing an initial situation that results from known outcomes. In scheme theory, reversible reasoning requires an additional level of interiorisation (Hackenberg, 2010; Ramful, 2014; Simon et al., 2016). We understand this to mean that students must have an anticipation (ability to anticipate the result scheme without carrying out the activity) in order to able to reverse the scheme. Through reversible reasoning, students can minimise the complexity of the problem, carry out an action in two opposite directions while being aware of the fact that it is the same action, and struggle to reverse the fact, concept, and relation.

We infer that reversible reasoning involves consequences of the past experience (making *activating*), explains the cause-effect connections to generate a particular cause (making *anticipating*), attempts to attain the goal by generating its cause (making *reversing*). Extending the work of Ramful (2014) and Hackenberg (2010) described reversible reasoning as sensitive to the numeric feature of problem parameter (e.g., proportional, fractional, and algebra situations). We establish that reversible reasoning could also be developed through an operational situation (e.g., derivative is the inverse operation of the antiderivative and vice versa, derivative of the antiderivative of a polynomial function is equal to the same function and vice versa) and relational situation (e.g., task requires obtaining information about f from a graph of f'). Haciomeroglu et al. (2009, 2010) indicated that reversible reasoning refers to the ability to establish bi-directional relationships, as opposed to one-way relationships that operate in one direction only. Consequently, we say that the students established a reversible reasoning if they recognised the relationship between the sign of the first derivative with the increase/decrease of the function, relationship between the sign of the first derivative with the concavity of function, and relationship between continuity with differentiability at a point.

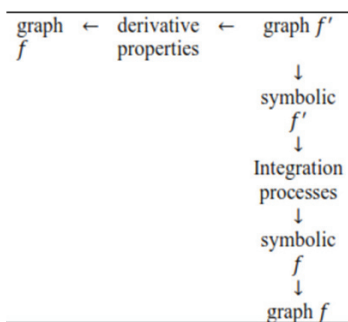
Flexibility in working with a graph of a derivative is essential for reversible reasoning in graphical situations (García et al., 2011; Tall, 2009). For example, the interpretation of the sign of the derivative function and another linked to increase/decrease derivative function and how students used conditional situations (Such as, *“if ... then ...”*). In addition, the reversible-reasoning type combines visual and analytical aspects to find a solution (García et al., 2011). Meanwhile, from the completion process perspective (Hong & Thomas, 2015), this type of thinker has an embodied a process of interval thinking

where they realise that graph f' is a gradient of the tangent line of curve f on a particular point or interval. These thinkers also exhibit the ability to identify a critical point, changes in the value of the gradient around the critical point to infer extreme values or turning points, and use statements “if $0 < x < 1$, $f'(x)$, is negative, graph f goes up or gradient f is positive within the interval”, “if $x = -1$ and $x = 1$, $f'(x)$, is zero, the gradient f in that point is flat or the gradient is 0”. In addition, most of these thinkers ignore information that f cannot be differentiated in $x = 0$ and form a singular point. This is because they fail to find the relationship between the continuity of graph f' and differentiable f on a certain point (Baker, Cooley, & Trigueros, 2000b; Cooley, 2002).

The third type of reasoner is those who combined direct-reasoning and reversible-reasoning in solving the problem. Hackenberg (2015) and Dominguez (2019) categorises reciprocal reasoning into reversible-reasoning. The subjects start with a direct process that is seen as a temporary answer and apply a reversible process to ensure the initial situation matches the interim answer. Mental activity can help individuals to solve complex problems by seeing the situation divergently. From another point of view (Gray & Tall, 2012), this thinking model includes conceptual encapsulation where two concepts are not seen as new.

The combination of direct-reasoning and reversible-reasoning lies in the scheme that has been formed, the emergence of an imbalance, the existence of mental actions by recalling the results of previous experiences and concepts that have been formed, and thinking about the causal relationship between problem situations.

Figure 7
Combined direct-reversible reasoning pathway



Specifically, the two types of thinking are needed in dealing with problems in mathematics, however, the reversible-reasoning type is more conceptual than direct-reasoning, because the subjects understand the causal relationship between functions and derivatives based on inherited traits and integration processes. This is in line with the opinion (Sangwin & Jones, 2017) that a problem that is reversible can encourage a problem-solver to conduct an in-depth analysis, forming the critical and creative thinker

characters. The inability of students to change the direction of their thinking from direct to reversible results from their previous experience that was focused on symbolic processes rather than visual processes (Thomas & Stewart, 2011) and most of the obstacles faced by the students are due to the dominance of this type of thinking. In addition, from the perspective of learning, reversible reasoning is rarely taken into account, and teachers are more prejudiced about the inability of their students to complete assignments by involving a reversible process.

Our data indicate three things: firstly, for the concept of derivative, the conscious use of the elements of knowledge exists when the student is able to apply the meanings and relationships between the mathematical elements of a function and the derivative function to the first derivative considered as a function and its derivative considering the information which comes from different modes of representation (García et al., 2011). Secondly, when students focus on derivative graph discontinuity at a point. They take a long time to understand their meaning and reflect on learned concepts. The notion of met-before is crucial for this condition when new learning is affected by the learner's previous experiences (Martin & Towers, 2016; McGowen & Tall, 2013). A met-before probably affect the mental structure that we have now resulting from experiences we have met before, for instance, making a relation between continuity and differentiability. Thirdly, another finding in this study is to characterise students' reversible reasoning. In our data, we observed that some students primarily focused on relationships between sign first derivative and increasing/decreasing of function at any x on an interval. In contrast, we found that other students primarily searching function formula on graphs, however, they experience obstacles in complex situations that result in re-analysis. In short, when students used relationships such as "if ... then ..." and the two forms of reversibility, we refer to their initial reversible. We use the term reversible to refer to reverse direction from the algebra to a relational perspective. Consequently, these two characteristics of reversible reasoning still require in-depth studies in further research.

Our findings contribute to the literature that exists on students' interpretations of graphs. We view our constructs as related to the works by Garcia et al. (2011) and Haciomeroglu et al. (2010) in describing students' interpretations of graphs as a whole. Reversible reasoning requires a justification of the relationship between the sign of the first derivative and the concavity of the function in a graphical context. From a cognitive perspective, the genetic decomposition of graphical interpretation of the derivative needs to be integrated with the learning process with technology (e.g., Geogebra) and allows other mental construction to emerge (Asiala et al., 1997; Aspinwall et al., 1997; Baker et al., 2000; Berry & Nyman, 2003; Cooley et al., 2007; Fuentealba et al., 2017, 2019; García-García & Dolores-Flores, 2019; Sánchez-Matamoros et al., 2019). For instance, students need the following genetic decomposition to solve a graphical task successfully: (1) prerequisite knowledge of a graphical representation of points, the concept of slope of a line, and the concept of function; (2) graphical interpretation of at a point and a function; (3) several coordination actions to get the graph original: graphical interpretation of $f(x)$ for a single x , interpretation of $f'(x)$ for a single x as the slope, process of moving through an interval (monotonic of the function and sign of the derivative, infinite slope

(vertical tangent) and infinite derivative, and concavity of the function and sign of the second derivative), and drawing a complete or fully representative graph.

Furthermore, the mathematical elements that the student used and the relationships that the student established between the elements to generate new information are required when students face reversible situation. Direct reasoning tends to a strong preference for algebra thinking. Here, students rely on symbolic representations and analytic thinking, but with they do not experience the need for visual thinking. In contrast, reversible reasoning tends to a strong synthesised visual and analytic thinking. They experience minimal different difficulties associated with their preferred modes for mathematical representation and thinking. Finally, in our research, we consider their conscious use to solve problems as a way of reasoning the different strengths of schema and how the reflective abstraction and the mental mechanism, and structure when processing information operate.

CONCLUSIONS

There are three types of mathematical reasoning involved when students are asked to seek the original graph from a derivative graph: direct reasoning, reversible reasoning, and combined direct-reversible reasoning. Direct reasoning is dominated by symbolic representations, so the students figured out the formula of f' through the integration process of obtaining f , found the intercept, and sketched the graphs. In contrast, reversible reasoning combines visual and analytic representations that coordinate condition related to graphical properties. This condition involves the meaning of derivative as the slope of the tangent line, at local graphic element and global graphic element, the relationship between values in f' with the up/down of graph f within an interval, the relationship between the up/down property of graph f' with the concavity of graph f , and the relationship between the discontinuity of f' and f indifference on a particular point. Students with direct reasoning have unstructured schema due to the unconsciousness use of the relations between a function and its derivative. In turn, students with reversible reasoning have a well-connected schema to respond graph of the derivative. Both reasoning patterns can be developed and need to be explored in future research by investigating them based on the assimilation accommodation, reflective abstraction, reification, or commognitive frameworks.

The results of this study can be used as a reference for further research in developing mathematical reasoning characteristics when sketching function graphs based on the derivative graphs, with different samples such as middle school students or pre-service teachers. In addition, the instruments used in this study can be used as a reference to reduce the direct reasoning tendency that is found in the majority of Indonesian students and to change the students' mental perception of mathematical problems that are not focused on symbolic representations, but on the coordination between two different representations (for example, visual-analytic). One of the goals of this study, although not specifically addressed here, is the enrichment of classroom instruction that identifies

and then challenges students' preferred modes of thinking; this goal is certainly worthy of future research.

Although the results of this study provide relevant information about which reasoning students needed to sketch the Original Graph from a Derivative Graph, we acknowledge that there are several limitations. Firstly, students' previous experiences and background may influence on how they attend to, interpret mathematical understanding, and make decisions. Secondly, we analysed students' written responses to examine how they make reasoning of what they read in students' answers. Differences reasoning may emerge if these students analysed information from analytical problems (e.g., sketching of original graph with several derivative properties). Thirdly, focusing on reversible reasoning could provide more rich information about students' thinking processes. Fourthly, reversible reasoning is also involved in other knowledge domains. For instance, students were constrained in a given set of visual representations of the graph showed $y = 2 \sin(x - \frac{\pi}{2})$, then they drew a graph starting from an algebraic representation. Students often have obstacles with restricted domains and require more considerations in an inverse situation (e.g., $\sin^{-1}(\sin \frac{\pi}{6}) = \frac{\pi}{6}$ however $\sin^{-1}(\sin \frac{7}{6}\pi) = -\frac{\pi}{6}$ because the domain of $\sin^{-1}(x)$ is restricted to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$). Finally, we have not examined the relationship between student mathematical knowledge or (resolution of problems) and how students retrieve or anticipate that knowledge. However, these limitations can inspire future research on how to develop typical students' direct and reversible reasoning in mathematical problems.

To date, the mathematics education literature has provided little theoretical guidance on promoting reversible reasoning (Ikram et al., 2020a, 2020b; Ramful, 2014; Simon et al., 2016). The analysis presented in this student case study involving a specific problem structure is bound to be narrow in focus. However, the moment-by-moment and fine-grained analysis of the interviews reveal aspects of students' thinking that may be informative and beneficial to teachers in their instructions on sketching the original graph from the derivative graph. The insights gained by analysing the ways in which they interacted with the tasks posed point to the necessity of promoting the interpretation of the sketch graph in instructions. Such an interpretation potentially allows students to use their knowledge of properties of derivative in graphical situations. The challenges that the participants experienced equally bring to the attention the necessity to foster relationships between the sign of the first derivative and the increase/decrease of the function, the sign of the second derivative and the concavity of the function, and continuity and differentiability as a form of reversible reasoning.

We consider important to underline that more research is needed to deeply understand reversible reasoning through the design of new instruments including tasks regarding the construction of relations between successive derivatives of higher order and analytical representation that involve value of the limit of function and derivative, continuity, first derivative, and second derivative. Also, it is important to stress that we take into account the construction of successive graph properties of derivatives when teaching Calculus, so that students are given opportunities to develop their mental action.

AUTHORS' CONTRIBUTIONS STATEMENTS

The four authors conceived the presented idea, developed the theory, and built the questionnaire, collected the questionnaire, and codified the data. All authors actively participated in the discussion of the results, reviewed, and approved the final version of the work

DATA AVAILABILITY STATEMENT

Data collected and analysed during the current study are available from the corresponding author, M.I, on reasonable request and at the authors' discretion.

ACKNOWLEDGMENTS

This research was funded by Lembaga Pengelola Dana Pendidikan (LPDP) Indonesia (grant no. 20161141081895).

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