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# Comparing the Didactic-Mathematical Knowledge of Derivative of In-Service and Pre-service Teachers

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## ABSTRACT

**Background**: The knowledge that a mathematics teacher should master has taken an increasing interest in recent years. Very few studies focused on comparing didactic-mathematic knowledge of in-service and pre-service teachers aimed at identifying features of the teachers' didactic-mathematical knowledge on specific topics that can establish a line between pre-service and in-service teachers' knowledge for teaching. **Objective**: The research aims to compare derivative knowledge of pre-service and in-service teachers to identify similarities and differences between teachers' knowledge. **Design**: This research is a mixed and interpretative study. **Settings and Participants**: The participants were 22 pre-service teachers, and 11 in-service teachers enrolled in a pre-service teacher education programme and a master's programme, respectively. **Data collection and participants**: Data were collected based on a questionnaire designed purposefully for the study. **Results**: The results show that pre-service teachers lack both epistemic derivative knowledge, while in-service teachers not only have this knowledge but relates it to its use in teaching. Pre-service teachers may not be making sense of the concept of derivative means, much less related to teaching. **Conclusions**: The insufficiencies found in pre-service teachers' knowledge justify the pertinence to design specific formative cycles to develop prospective teachers' epistemic facet of didactic-mathematical knowledge. It is recommended that both in-service and pre-service teachers discuss activities in which they can identify and reflect on possible mistakes and errors made by students. The development of these formative cycles should consider the complexity of the global meaning of the derivative.

**Keywords**: teacher knowledge, derivative, didactic-mathematical knowledge, questionnaire, teachers.

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**Uma comparação do conhecimento didático-matemático**

**sobre a derivada de professores em exercício e futuros professores**


## RESUMO

**Antecedentes**: O conhecimento que um professor de matemática deve dominar tem despertado um interesse crescente nos últimos anos. No entanto, há muitas poucas investigações centradas na comparação dos conhecimentos didático-matemáticos dos professores em exercício e dos professores em formação inicial, destinadas a identificar características dos conhecimentos didático-matemáticos dos professores sobre temas específicos, que possam estabelecer uma linha de fronteira entre os conhecimentos didático-matemáticos dos professores em formação inicial e os conhecimentos dos professores em exercício para o ensino. **Objetivo**: O objetivo é comparar conhecimentos da derivada dos professores em exercício e dos professores em formação inicial. **Design**: Esta pesquisa é um estudo misto e interpretativo. **Ambiente e participantes**: Os participantes são 22 professores de formação inicial e 11 professores em exercício, inscritos em um programa de treinamento de professores e em um programa de mestrado, respectivamente. **Coleta e análise de dados**: Os dados foram coletados com base em um questionário elaborado propositadamente para o estudo. **Resultados**: Os resultados evidenciaram o fato de os professores de formação inicial não terem os conhecimentos sobre a derivada, enquanto os professores em exercício não só têm esses conhecimentos, mas, além disso, os relacionam com a sua possível utilização no ensino. **Conclusões**: As insuficiências nos conhecimentos dos professores de formação inicial justificam a pertinência de ações formativas específicas, a fim de desenvolver a faceta epistémica do conhecimento didático-matemático dos futuros professores. Recomenda-se que tanto os professores em exercício quanto os professores de formação inicial discutam atividades em que tenham a oportunidade de identificar e refletir sobre possíveis erros cometidos por alunos ou colegas. O desenvolvimento destes ciclos formativos deve considerar a complexidade do significado global da derivada.



**Palavras-chave**: conhecimento dos professores, derivada, conhecimento didático-matemático, questionário, professores.


## INTRODUCTION

Teacher education is experiencing unprecedented attention by the mathematics education research community and the government. Colombian Ministry of Education has recently regulated the teacher education programmes to meet high expectations for Colombian mathematics education. According to Sharplin, Peden, and Ida Marais (2016), “the introduction of a teacher standard framework is consistent with international trends in teacher quality policy” (p. 1). Mathematical and didactic training of mathematics teachers is a research field that claims the attention of the research community on educational mathematics and educational administrations alike. The main reason is that students' mathematical competencies essentially depend on teachers' education. The derivative is one of the fundamental concepts covered in calculus; it is epistemologically difficult for students (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Furinghetti & Paola, 1991; Hauger, 2000). It is reported that students can give correctly "the slope of the tangent line at a given point on a graph" definition of a derivative; nonetheless, they make wrong interpretations of this definition (Amit & Vinner, 1990; Ubuz, 2001). In addition, students have problems conceptualising and relating the rate of change to the concept of derivative (Bezuidenhout, 1998; Heid, 1988; Orton, 1983). Another conceptual difficulty is noticing the difference between the average rate of change and the instantaneous rate of change in relating these concepts to derivative (Bingölbali & Monaghan, 2008; Orton, 1983).

According to Sahin, Yemmez & Erbas (2015), students find it difficult to conceptualise the role of limit in (i) providing an algebraic definition of derivative, (ii) understanding how the average rate of change approximates to the instantaneous rate of change and, (iii) understanding how the slopes of the secant lines approximate to the slope of the tangent line (Hankiöniemi, 2006; Orton, 1983).

There is a great deal of research on students' difficulties with the concept of derivative (Orhun, 2012). Still, there are not so many studies that deal with the knowledge of both pre-service teachers and in-service teachers, concerning knowledge of the derivative and its teaching (Sánchez-Matamoros, Fernández, & Llinares, 2014; An & Wu, 2012; García, Llinares, & Sánchez-Matamoros, 2011).

Looking at teachers' teaching didactic knowledge of derivative and how that knowledge would affect their ability to solve problems can contribute to our understanding of both pre-service and in-service teachers' skills and conceptual understanding and generate insights into their thinking about such knowledge, suggesting ways to improve it.

One of the most pressing issues is to know the required didactical-mathematical knowledge for teaching mathematics. Careful considerations and recommendations of research (Shulman, 1986; Ball, 2000; Ball, Lubiensky & Mewborn, 2001; Hill, Ball, & Schilling, 2008) suppose some progress in the components' characterisation on the knowledge that a teacher must have to develop an efficient practice and promote the student's learning process. However, as Godino (2009) points out, the mathematical knowledge models for teaching developed by researchers on mathematics education include very global categories; therefore, it would be interesting to use models that allow for a more detailed analysis of each one of the knowledge types brought into play for an effective mathematics teaching. Besides, it is necessary to focus on specific topics, such as the high school teacher's required knowledge to teach the derivative (García, Azcárate, & Moreno, 2006; Badillo, Azcárate & Font, 2011). The complexity of demands faced by pre-service and in-service teachers to perform their profession is increasing over time. Thus, the comparison between pre-service and in-service teachers' didactic-mathematical knowledge could bridge the frequently manifest gap between university and school in pre-service teachers' professional anticipation. This article aims to compare in-service and pre-service teachers' knowledge to identify key features differentiating and characterising their knowledge to propose education cycles to improve in-service and pre-service teachers' knowledge.

In the next section, we will present the main ideas of the teachers' model knowledge. Then, we will have the methodology and bring the results and discussions. Finally, we present our final reflexions.

## THEORETICAL BACKGROUND

In mathematics education research, several models try to identify and describe features that integrate the teachers' knowledge required to teach mathematics. Shulman (1986, 1987), Grossman (1990), Ball and colleagues (Ball, Lubienski, & Mewborn, 2001; Hill, Ball, & Schlling, 2008), Rowland, Huckstep and Thwaites (2005), and Schoenfeld and Kilpatrick (2008) wrote articles that seek to characterise such knowledge. Nonetheless, there are some limitations in these models, as stated by Silverman and Thompson (2008),

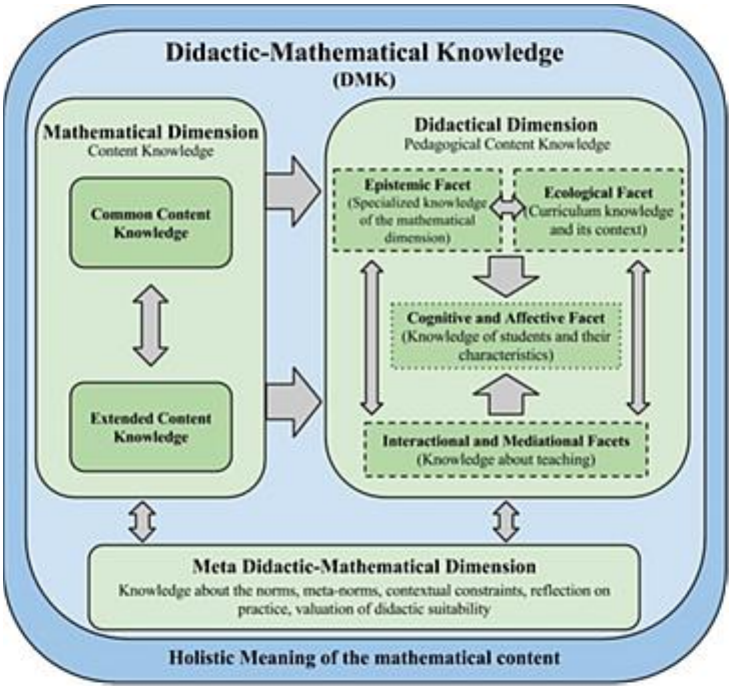
> While the mathematical knowledge for teaching has started to gain attention as an important concept in the mathematics teacher education research community, there is limited understanding of what it is, how one might recognize it, and how it might develop in the minds of teachers. (p. 499)

In this paper, we used the didactic-mathematical knowledge (DMK) model, which draws upon theoretical assumptions and theoretical-methodological tools of the theoretical framework known as the onto-semiotic approach (OSA) to mathematical cognition and instruction (Godino, Batanero & Font, 2007). The DMK model takes into consideration: (1) the contribution and development of the theoretical framework OSA, which has been developed in several research studies by Godino and colleagues (Godino, Batanero & Font, 2007; Font, Godino & Gallardo, 2013); (2) the findings and contributions proposed by authors such as Shulman (1986, 1987), Grossman (1990), Hill, Ball and Schlling (2008), Schoenfeld and Kilpatrick (2008), Rowland, Huckstep and Thwaites (2005); and (3) the results obtained in several empirical studies (e.g., Pino-Fan, Godino & Font, 2013; Pino-Fan, Assis & Castro, 2015; Castro, Pino-Fan & Velasquez-Echavarría, 2018).

The DMK model interprets and characterises the teacher's knowledge from three dimensions (Figure 1): mathematical dimension, didactical dimension, and meta-didactic mathematical dimension.

**Figure 1**

*Dimensions and components of the DMK.* Taken from Pino-Fan, Godino, & Font (2018, p. 66).




DMK's mathematical dimension refers to the knowledge that a teacher needs to teach mathematic or guide classroom mathematical activity. The model includes two subcategories of knowledge: common content knowledge and extended content knowledge. The first subcategory, common content knowledge, is the knowledge that is considered sufficient to solve problems and tasks proposed in the school mathematics textbooks; it is a shared knowledge by teachers and the students. Extended content knowledge refers to the knowledge required to suggest new mathematical challenges in the classroom, to link mathematical objects under study and to guide students to the study of subsequent mathematical notions to be found in curriculum and in daily life. These two subcategories are reinterpretations of both the common content knowledge (Hill, Ball, & Schlling, 2008) and the horizon knowledge (Ball & Bass, 2009), respectively. Pino-Fan, Assis, and Castro (2015, p. 1434-1436) point out that the Epistemic facet refers to specialised knowledge of the mathematical dimension.

The teacher must have not only mathematics knowledge to solve mathematics problems, but also mathematical knowledge 'shaped' for teaching; that is to say, the teacher must mobilise mathematical object's representations to solve mathematics tasks according to students' previous knowledge, linking mathematical objects located in curriculum, providing explanations according to pupils' doubts and contexts.

## METHODOLOGY

The methodological approach used is of mixed type (Johnson & Onwuegbuzie, 2004) because it involves an exploratory-type study in which the observations of quantitative features are considered (degree of correctness of items: correct, partially correct, and incorrect answers) and qualitative features (solution type or cognitive configurations proposed by teachers). The qualitative features are closely related to the type of didactical-mathematical knowledge concerning the derivative of both prospective and in-service high school teachers.

### Individuals and Context

The test was applied to three groups of informants: 11 students enrolled in the sixth semester of the bachelor's degree in Mathematics Teaching, in the School of Education; 11 students enrolled in the School of Sciences, both at the University of Antioquia, Medellin, Colombia and 11 postgraduate students, enrolled in a masters' programme of didactic of mathematics, in a Colombian university. For this report, we do not distinguish the first two groups[1].

All 22 prospective teachers, to whom the questionnaires were applied, have coursed differential calculus for their bachelor's degree. All 22 students took courses such as integral calculus, vector calculus, and differential equations. The students enrolled in the School of Education took other subjects related to mathematics and its didactics. Those enrolled in the School of Sciences have taken none of them.

**The Questionnaire**

The questionnaire explores the content knowledge about the derivative and the teaching knowledge concerning teacher assessment of incorrect students' solutions. Eight tasks were designed that brought into play different types of representations concerning these three processes: verbal description, graphic, formula (symbolic) and tabular; for both the function and its derivative (epistemic facet of DMK). Seven of these tasks were taken from a questionnaire designed in the doctoral thesis written by Pino- Fan (2014), while the last one was designed expressly for this report. Nonetheless, all of them were submitted to researchers in the field to validate the content, the construct, and the ecology.

The questionnaire focuses on assessing partial aspects of the mathematical dimension of the DMK of prospective and in-service high school teachers concerning the derivative. Such facet, according to the DMK model (Pino-Fan, Assis & Castro, 2015; Pino-Fan, Godino, & Font, 2018; Castro, Pino-Fan, & Velásquez-Echavarría, 2018), includes three types of knowledge: common content knowledge, extended content knowledge, and the epistemic facet.

We consider that didactic-mathematical knowledge includes three types of tasks that require: (1) bring into play the common content knowledge (to solve the mathematical task proper of the high school mathematics); (2) the epistemic didactic-mathematical knowledge (use different representations, different partial meanings of a mathematical object, solve the problem through different procedures, give several relevant arguments, identify the knowledge

[1] All ethical protocols of informed consent by the participants were used in the research. The authors explicitly exempt *Acta Scientiae* for any consequences arising therefrom, including full assistance and possible compensation for any damage resulting to any of the research participants, in accordance with Resolution No. 510, of April 7, 2016, of the National Health Council of Brazil.

brought into play during the resolution of a mathematical task, etc.); and (3) a teacher requires the extended content knowledge (to generalise tasks about common or epistemic knowledge and perform connections with more advanced mathematical objects in the curriculum). It includes questions to test the knowledge that teachers must identify meaning conflicts between personal knowledge and institutional knowledge (Godino, Batanero, & Font, 2007).

**Tasks**

Each one of the tasks included in the questionnaire and the analysis of the aspects which are evaluated by each one of them is presented in what follows. Task 1 is a classic question that has been used in different research works (Badillo, 2003; Hähkiöniemi, 2006; Habre & Abboud, 2006; Bingolbali & Monaghan, 2008) to explore the meanings known by students about the derivative. Being a global question, teachers[2] are expected to give 'lists' of possible derivative meanings. For these reasons, this task explores the teachers' knowledge related to the meanings of a derivative.

Task 2 has been discussed in several papers (Tsamir, Rasslan, & Dreyfus, 2006; Santi, 2011) that search for the three types of knowledge that constitute the epistemic facet of didactic-mathematical knowledge concerning the derivative: 1) common content knowledge (Item a), as the teacher should solve the Item using several representations or argumentations; 2) specialised content knowledge (items b, c, and e) because, apart from solving the items, they require the use of representations by teachers (graphic, symbolic, and verbal) and argumentations which justify their procedures; and 3) extended content knowledge (item d), because it requires teachers to generalise the initial task about the derivability of the absolute value function at x=0, including valid justifications for the proposition "the graph of a derivable function cannot have peaks," the definition of the derivative as an instantaneous rate of variation (limit of the quotient of increments). The meanings of the derivative as the slope of the tangent line and instantaneous rate of variation are associated with this task.

Task 3, taken from Delos Santos (2006) and Pino-Fan (2014), explores the teachers' extended content knowledge because its solution requires the use of more advanced mathematical objects in the high school mathematics curriculum, such as the integral of a function or the fundamental theorem of calculus. The representations that a respondent should use for a task solution are symbolic, graphic, and tabular. The extended content knowledge assessed

[2] Pre-service teachers and in-service teachers.

in this task is associated with the meaning of the derivative as the slope of a tangent line.

Task 4, discussed by Viholainen (2008), inquiries about the epistemic facet of DMK; its solution requires different representations (graphic, verbal description, formula) and justifications to explain the sentence ‘the derivative of a constant function is always zero.’ Several meanings associate with the derivative: the slope of the tangent line, instantaneous rate of change, and instantaneous rate of variation can be considered.

Task 5 could be seen as one of the exercises commonly found in high school differential calculus books, in which some theorems or propositions can be applied concerning derivatives for their solving. For this reason, both Item a) and item b), separately, evaluate common content knowledge features related to the derivative in its meaning as the slope of the tangent line and instantaneous rate of change. However, the primary objective of the task is to explore the mathematical activity developed by teachers globally and whether they can establish connections among different derivative meanings. In this sense, Task 5 assesses aspects of the epistemic facet of the DMK while seeking the association done by teachers among different derivative meanings.

Task 6 was taken and modified from Çetin (2009). Both items a) and item b) yield information about the epistemic facet of the DMK, related to the derivative meaning as the instantaneous rate of change. On one side, Item a) requires an interpretation concerning the verbal-linguistic, graphic elements (graphics of derivatives), and iconic (images of cups) to attempt establishing an injective correspondence between graphic and iconic elements. Next, respondents should find procedures allowing establishing the correlation between each element and giving valid justifications to their solutions.

Looking for such procedures, teachers must use mathematical objects such as the derivative as instantaneous rate of filling of a container (filling velocity), increasing and decreasing of functions, fundamental theorem of calculus. They should also be able to move through different representations and change the natural language to express their results. However, item b) requires identifying knowledge (linguistic elements, concepts/definitions, properties/propositions, procedures, and arguments) that are brought into play at task solving; its objective is to manage prospective pupils’ knowledge efficiently. Thus, Task 6 is the tester of two levels of the epistemic facet of the DMK; the first level, in which teachers should use several representations, concepts, propositions, procedures, and arguments to solve the task, and the second level refers to the knowledge that the teacher requires to identify the elements that make up the mathematical practice of their future students to solve a derivative task.

Task 7 gives information about teachers' extended content knowledge, because it deals with the derivative approximation to a function (described by the values of the table) at point t=0.4, through numerical values of such function. Besides, Task 7 is not a typical school problem at the high school level and requires teachers to understand the derivative, at least its meaning as an instantaneous rate of change, and specifically the derivative as instantaneous velocity. The solution to this task can be done by different methods, for example, Lagrange's polynomial interpolation; this supports the categorisation of this task as the tester of the extended content knowledge.

Task 8 refers to both the teachers' assessment of pupils' answers and student's comprehension of meaning conflicts. The question is about the teachers' assessment of a quite common student's answer. The error may be related to the mnemotechnic strategies used by students. Mnemonic instruction has been empirically validated as a technique that can enhance students' learning (Berkeley, Scruggs & Mastropieri, 2010; Carney & Levin, 2000). Mnemonic instruction has been documented to be versatile as it can be effectively used not only across abilities but across subject areas, including foreign language, English, science, history, mathematics, and social studies (Letendre, 1993; Scruggs, Mastropieri, Berkeley, & Graetz, 2009; Brigham, Scruggs, & Mastropieri, 2011), but sometimes the techniques applied without due care can lead to errors.

## RESULTS AND DISCUSSION

The analysis of data considers the degree of task's correctness (correct, partially correct, and incorrect answers) and type of cognitive configuration (resolution by the pre-service and in-service teachers, specifying objects and process brought into play). The analysis of data concerning the cognitive configuration is carried out by the technique known as semiotic analysis (e.g., Malaspina & Font, 2010; Godino, Font, Wilhelmi, & Lurduy, 2011), which allows describing systematically both the mathematical activity performed by teachers while the primary mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures, and arguments) are put in place to solve the problems.

The type of didactic-mathematical knowledge is closely related to the feature type of cognitive configuration associated with students' answers, because the epistemic facet of didactic-mathematical knowledge depends on the presence or absence of the mathematical objects, their meanings, and relations among them. The cognitive configurations have a didactic-mathematical nature due to the displayed tasks of the same nature, and therefore the subjects should handle the didactic and mathematical knowledge.

Concerning the degree of correctness, punctuations 2, 1 or 0 were assigned if the answers were correct, partially correct, or incorrect, correspondingly. We will show the results in two separate tables, one for pre-service teachers and another for in-service teachers.

**Results for Task 1: Meanings of the derivative**

Table 1 shows the results for Task 1 concerning the features degree of correctness and type of cognitive configuration. Table 2 shows the information for in-service teachers. An answer is considered correct if the respondent uses at least one partial meaning of the derivative.

**Table 1**

*Degree of correction and meanings of the derivative: Pre-service teachers*

| Level of accuracy | Task 1 | | Meanings of the derivative | Task 1 | |
|---|---|---|---|---|---|
| | Frequency | % | | Frequency | % |
| **Correct** | 18 | 81.8 | **Slope of the tangent line** | 10 | 45.4 |
| **Incorrect** | 2 | 9.2 | **Instantaneous rate of change** | 6 | 27.2 |
| **No answer** | 2 | 9.2 | **Instantaneous rate of variation** | 2 | 9.0 |
| **Total** | 22 | 100 | **Two meanings** | 2 | 9.0 |
| | | | **Three meanings** | 1 | 4.5 |
| | | | **Other** | 1 | 4.5 |
| | | | **No solution** | 0 | 0 |
| | | | **Total** | 22 | 100 |

Since the question is ‘global,’ it intends to explore derivative meanings known by respondents. Table 1 shows that, in general, pre-service teachers did not have problems answering the task. 81.8% of them gave a correct answer to it. All of them assigned a meaning to the derivative. Most students (10) corresponding to the 45.4% assigned the meaning ‘slope of the tangent line’, and 27.2% assigned the meaning ‘instantaneous rate of change.’ Table 2 shows the same pattern for in-service teachers, who choose as the main meaning for the derivative ‘slope of the tangent line’, and the next chosen meaning was ‘instantaneous rate of change.’ Interestingly, even though 36.2 % of the pre-service teachers chose ‘instantaneous rate’ as one of the derivative meanings, only 18.3% use the concept to solve Task 5b), which require finding where the instantaneous rate of change is zero. According to Desfriti (2016), none of the pre-service teachers in her study could provide a comprehensive explanation of limit or derivative, “most of them just rewrite definitions as they were provided in textbooks. Several were able to explain and gave related examples” (p.5). The mathematics knowledge expressed by teachers is formal and located in the common content knowledge that is a basic knowledge from which the specialised content knowledge derives.

**Table 2**

*Degree of correction and meanings of the derivative: In-service teachers*

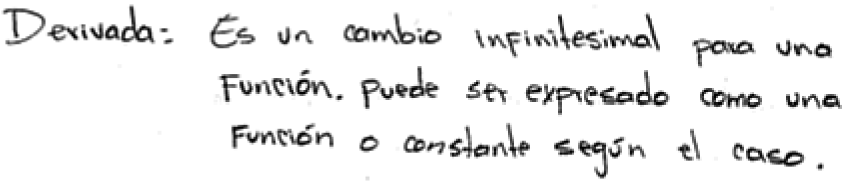
| Level of accuracy | Task 1 | | Meanings of the derivative | Task 1 | |
|---|---|---|---|---|---|
| | Frequency | % | | Frequency | % |
| **Correct** | 11 | 100 | **Slope of the tangent line** | 5 | 45.4 |
| **Incorrect** | 0 | 0 | **Instantaneous rate of change** | 2 | 18.1 |
| **No answer** | 0 | 0 | **Instantaneous rate of variation** | 1 | 9.09 |
| **Total** | 11 | 100 | **Two meanings** | 1 | 9.09 |
| | | | **Three meanings** | 1 | 9.09 |
| | | | **Other** | 1 | 9.09 |
| | | | **No solution** | 0 | 0 |
| | | | **Total** | 11 | 100 |

Figure 2 displays two prototypical examples of pre-service teachers’ answers which have been labelled as ‘other’ meanings. Other meanings given to the derivative in this category are: “it is a function”; “it is a procedure”; “better linear approximation”; “converse process to the integral”.

**Figure 2**

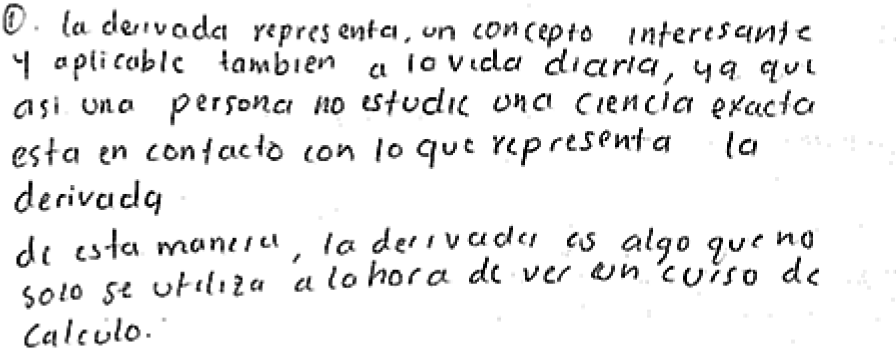
*Other meanings given by pre-service teachers concerning the derivative*

Answer 1


Derivada: Es un cambio infinitesimal para una Función. Puede ser expresado como una Función o constante según el caso.

*[The derivative] is an infinitesimal change to a function. It can be expressed as a function or a constant.*

Answer 2


①. la derivada representa, un concepto interesante y aplicable tambien a lo vida diaria, ya que asi una persona no estudie una ciencia exacta esta en contacto con lo que representa la derivada
de esta manera, la derivada es algo que no solo se utiliza a la hora de ver un curso de Calculo.

*The derivative represents, an interesting and applicable concept to daily life, no matter is someone does not study a science, it is in contact with what represents the derivative in this way, the derivative is something that is not only used when taking a calculus course.*

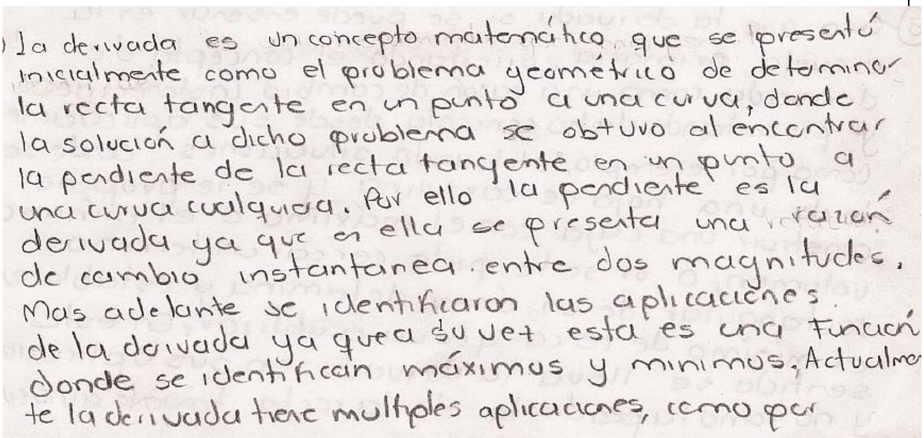
The study conducted by Gökçek and Açıkyıldız (2016) showed that candidate teachers have superficial knowledge about derivative concept, “since candidate teachers, whose understandings were mainly restricted to definitions, they could not fully consolidate the content of definitions” (p. 130). The meanings proposed by pre-service teachers are behind what could be considered as "right mathematical definitions" to be used during classes, the common content knowledge manifested by pre-service teachers are not strong enough to be put into play neither to teach nor to solve mathematics routine tasks.

Figure 3 shows an in-service teacher' answer, and doubtless to say that it is more elaborated that the answer shown in Figure 2.

**Figure 3**

*Other meanings given by in-service teachers concerning to the derivative*

Answer


1) La derivada es un concepto matemático que se presentó inicialmente como el problema geométrico de determinar la recta tangente en un punto a una curva, donde la solución a dicho problema se obtuvo al encontrar la pendiente de la recta tangente en un punto a una curva cualquiera. Por ello la pendiente es la derivada ya que en ella se presenta una razón de cambio instantanea entre dos magnitudes. Mas adelante se identificaron las aplicaciones de la derivada ya que a su vez esta es una Función donde se identifican máximos y minimos, Actualmente la derivada tiene multiples aplicaciones, como por

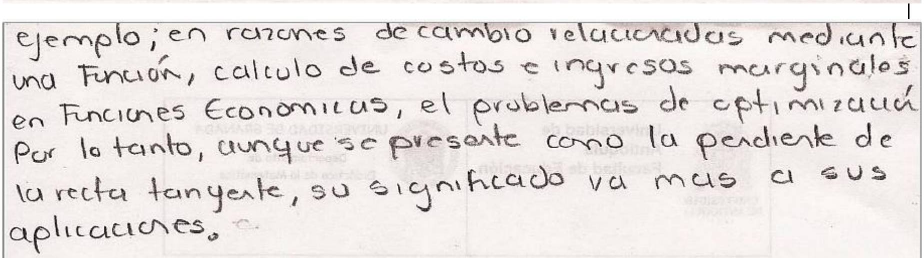

ejemplo; en razones de cambio relacionadas mediante una Función, calculo de costos e ingresos marginales en Funciones Economicas, el problemas de optimización. Por lo tanto, aunque se presente como la pendiente de la recta tangente, su significado va mas a sus aplicaciones.

**Figure 3 (Cont'd)**

*The derivative is a mathematical concept that was presented as a geometrical problem to find the tangent line to a given point to any function. That is why the slope is the derivative because in it the instantaneous rate of change between two magnitudes, further the derivative applications were identified because the derivative itself is a function where maxima are* *identified. Currently, the derivative has many applications, for instance; in the related rate of change using a function, computing costs, and marginal incomes, in economic functions, in optimization problems. So, even though it is presented as the slope of a tangent line, its meaning is applied more to applications.*

By contrast, the meanings expressed by in-service teachers are suited to be used in real mathematics classes as they are not only correct but expressed in colloquial class-like terms. Otero and Llanos (2019) concluded from their study that the questions "what is" posed to in-service teachers promoted closed answers as definitions and revealed that teachers conceive mathematics as immutable.

**Results for Task 2: Derivative of the absolute-value function**

Table 3 shows the results obtained, by pre-service teachers, in Task 2, regarding the variable degree of correctness. Items b) and c) are related to common knowledge and aspects of the epistemic facet of the DMK, and d) is related to the extended knowledge, and all that was more difficult for the pre-service teachers.

Considering the partial, correct, and incorrect answers concerning item b), we can see that 63.6 % of prospective teachers had problems solving it. Concerning item c), considering the partially correct, incorrect, and no answered questions, 45.4 % of prospective teachers had difficulties giving a satisfactory answer. The above reveals that more than half of the prospective teachers show deficiencies concerning the common and epistemic facet knowledge of the required content for solving the task.

Regarding item d) five students (22.7%) could generalise the task for any function with 'peaks', and three (13.6 %) gave approximations to such generalisation without finishing it. This result shows that more than half of the teachers have a scarce required extended knowledge for providing a solution to the task. Table 4 shows the results obtained, by in-service teachers, from Task 2.

**Table 3**

*Frequencies and percentages for Task 2: Pre-service teachers*

| Level of accuracy | Section a) | | Section b) | | Section c) | | Section d) | |
|---|---|---|---|---|---|---|---|---|
| | Frequency | % | Frequency | % | Frequency | % | Frequency | % |
| **Correct** | 14 | 63.6 | 11 | 50 | 12 | 54.5 | 5 | 22.7 |
| **Partially correct** | 0 | 0 | 7 | 31.8 | 3 | 13.6 | 3 | 13.6 |
| **Incorrect** | 6 | 27.2 | 3 | 13.6 | 5 | 22.72 | 3 | 13.6 |
| **No answer** | 2 | 9.09 | 1 | 4.5 | 2 | 9.0 | 11 | 50 |
| **Total** | 22 | 100 | 22 | 100 | 22 | 100 | 22 | 100 |

**Table 4**

*Frequencies and percentages for the correction level of task 2: In-service teachers*

| Level of accuracy | Section a) | | Section b) | | Section c) | | Section d) | |
|---|---|---|---|---|---|---|---|---|
| | Frequency | % | Frequency | % | Frequency | % | Frequency | % |
| **Correct** | 11 | 100 | 11 | 100 | 11 | 100 | 11 | 0 |
| **Partially correct** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **Incorrect** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **No answer** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **Total** | 11 | 100 | 11 | 100 | 11 | 100 | 11 | 100 |

All in-service teachers did very well on this task. Concerning the types of solutions, we find three types of resolutions that respondents use to solve Task 2; each one of them connected with a configuration of objects and meanings. These three types of cognitive configuration are denominated graphic verbal (use of graphics and explanations), technical (use), and formal (demonstrates that the limits in zero do not exist). Table 5 shows that a high percentage of prospective teachers proposed solutions with a graphic-verbal configuration for sections a) and c) (e.g., "…is not derivable in x = 0, since infinite tangents to the function can be drawn at that point"). For section b), most of the pre-service teachers provide a technical configuration (using derivation rules and the definition of absolute value). Four students (18.18%) provided a formal solution from the meaning of the derivative as the instantaneous rate of variation in the four sections of the task, and one respondent (4.5%) provided a formal configuration for section c).

**Table 5**

*Type of cognitive configuration of Task 2: Pre-service teachers*

| Cognitive Configuration | Section a) | | Section b) | | Section c) | |
|---|---|---|---|---|---|---|
| | Frequency | % | Frequency | % | Frequency | % |
| **Graphic-Verbal** | 16 | 72.7 | 8 | 36.3 | 15 | 68.1 |
| **Technical** | 2 | 9.0 | 9 | 40.9 | 6 | 27.2 |
| **Formal** | 4 | 18.18 | 5 | 22.7 | 1 | 4.5 |
| **No solution** | 0 | 0 | 0 | 0 | 0 | 0 |
| **Total** | 22 | 100 | 22 | 100 | 22 | 100 |

Table 6 shows that all in-service teachers offered a graphic-verbal solution to section a). Sections a) and c) (e.g., "…is not derivable in x = 0, since infinite tangents to the function can be drawn at that point"), it seems a little strange that all teachers, in-service and pre-service, choose the same sort of meaning to answer Item a). For section b), most in-service teachers provide a technical configuration (using derivation rules and the definition of absolute value). One in-service teacher provided a formal solution for the derivative in x=2, while five teachers choose to give graphic and verbal solution to the derivative at zero.

**Table 6**

*Type of cognitive configuration of Task 2: In-service teachers*

| Cognitive Configuration | Section a) | | Section b) | | Section c) | |
|---|---|---|---|---|---|---|
| | Frequency | % | Frequency | % | Frequency | % |
| **Graphic-Verbal** | 11 | 100 | 3 | 27.2 | 5 | 45.4 |
| **Technical** | 0 | 0 | 7 | 63.6 | 2 | 18.1 |
| **Formal** | 0 | 0 | 1 | 9.0 | 4 | 36.3 |

| | | | | | | |
|---|---|---|---|---|---|---|
| **No solution** | 0 | 0 | 0 | 0 | 0 | 0 |
| **Total** | 11 | 100 | 11 | 100 | 11 | 100 |

In their study, Tsamir, Rasslan, and Dreyfus (2006) report that "the students correctly defined the derivative of a function at a given point. That is, their concept definition seemed satisfactory" (p.248); however, they did not use their knowledge of the definitions, revealing concept images that are sometimes incompatible with their definitions. This is the case of some teachers; for instance, one of them wrote, "The derivative represents an interesting and applicable concept to daily life..." (Figure 2), what, apart from being correct, is the kind of content knowledge that a pre-service teacher should have to explain it to students. What this pre-service teacher stated is a kind of declarative knowledge, nor true nor false but certainly not suited to be used in a real-life calculus class. Pino-Fan et al. (2013) used the same task in their research conducted with Mexican students and reported that a high percentage of prospective teachers, 88.6% and 54.7%, respectively, provided a graphic-verbal configuration to items a) and c) (e.g., "…it is not derivable at x=0, because an infinite number of tangent lines to the function can be traced on that point. p. 3200). Their results resemble ours regarding that the prospective teachers manifest difficulties to solve tasks related not only to the specialised and extended content knowledge but also with the common content knowledge. Seldem, Mason, and Selden (1989) informed that students could not solve non-routine calculus problems in their study.

### Results for Task 3: Calculating a primitive function

Table 7 shows the results for Task 3 concerning the degree of correctness. It seems that pre-service teachers had no trouble solving Item a) of the task, 63.3% answered it correctly, but they had difficulties solving item b), while 18.8 % gave an incorrect answer, and 18.8 % did not answer.

**Table 7**

*Degree of correction of Task 3: Pre-service teachers*

| Level of accuracy | Section a) | | Section b) | |
|---|---|---|---|---|
| | **Frequency** | **%** | **Frequency** | **%** |
| **Correct** | 14 | 63.3 | 8 | 36.6 |
| **Partially correct** | 0 | 0 | 6 | 27.2 |

| | | | | |
|---|---|---|---|---|
| **Incorrect** | 3 | 13.2 | 4 | 18.8 |
| **No answer** | 5 | 22.7 | 4 | 18.8 |
| **Total** | 22 | 100 | 22 | 100 |

Table 8 shows the results for Task 3 concerning the degree of correctness for in-service teachers. It seems that pre-service teachers had no trouble solving either Item a) or item b) of the task. It seems that the knowledge teachers put into play to solve mathematics tasks are located at various levels of development, while all in-service teachers responded to Item a), it was not the case for item b). Sometimes, the knowledge is in place, but relationships among them seem to be missing.

Table 9 shows the frequencies and percentages for the type of cognitive configurations used by pre-service teachers to solve the task. The types of cognitive configurations for Item a) Task 3 are: The Graphic–Technical refers to the use of graphic characteristics to support the solution; Numerical-Technical refers to the use of numerical instances to support the solution; Graphic-Advance uses families of functions to illustrate the graphic meaning of the constant in representing $f(x) = x^2 + c$; Numerical-Advanced refers to the use of the constant function but referring to it as any number. The types of cognitive configuration for Item b) Task 3 are: Advance, which refers to the use of the unicity of the derivative and the null derivative of any constant function; Technical, which states that there is only one function whose derivative is 2x; Erroneous Uniqueness states that students say that "there is no other function, because I could not find it." Akkoç, Bingolbali, and Ozmantar (2008) report that teachers could not relate the notion of rate of change to graphical or numerical meaning of derivative.

**Table 8**

*Degree of correction of Task 3: In-service teachers*

| **Level of accuracy** | **Section a)** | | **Section b)** | |
|---|---|---|---|---|
| | **Frequency** | **%** | **Frequency** | **%** |
| **Correct** | 11 | 100 | 11 | 100 |
| **Partially correct** | 0 | 0 | 0 | 0 |
| **Incorrect** | 0 | 0 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| **No answer** | 0 | 0 | 0 | 0 |
| **Total** | 11 | 100 | 11 | 100 |

**Table 9**

*Type of cognitive configuration of Task 3: Pre-service teachers*

| **Cognitive configuration** | **Section a)** | | **Cognitive configuration** | **Section b)** | |
|---|---|---|---|---|---|
| | **Frequency** | **%** | | **Frequency** | **%** |
| **Graphic-Technical** | 2 | 9.0 | Advanced | 1 | 4.5 |
| **Numerical-Technical** | 20 | 90.9 | Technical | 19 | 86.3 |
| **Graphic-Advanced** | 0 | 0 | Erroneous Uniqueness | 2 | 9.09 |
| **Numerical-Advanced** | 0 | 0 | Equivalent functions | 0 | 0 |
| **No solution** | 0 | 0 | No solution | 0 | 0 |
| **Total** | 22 | 100 | Total | 22 | 100 |

Table 10 shows the results for Task 3 concerning degree of correctness for in-service teachers. It seems that in-service teachers had no trouble solving either Item a) or Item b) of the task. By comparing the pre-service and in-service solutions, we can note that they used the same kind of cognitive configuration. The only difference was in the "Erroneous Uniqueness" configuration, which was not used by in-service teachers.

**Table 10**

*Type of cognitive configuration of Task 3: In-service teachers*

| **Cognitive configuration** | **Section a)** | | **Cognitive configuration** | **Section b)** | |
|---|---|---|---|---|---|
| | **Frequency** | **%** | | **Frequency** | **%** |
| **Graphic-Technical** | 3 | 27.2 | Advanced | 7 | 63.6 |

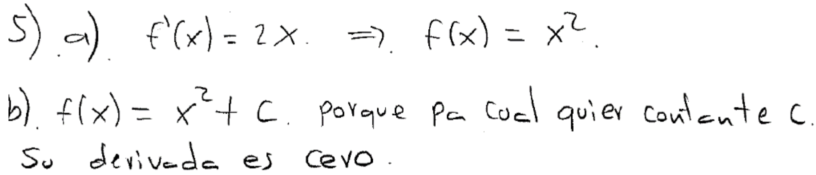
| | | | | | |
|---|---|---|---|---|---|
| **Numerical-Technical** | 3 | 27.2 | Technical | 2 | 18.1 |
| **Graphic-Advanced** | 2 | 18.1 | Erroneous Uniqueness | 0 | 0 |
| **Numerical-Advanced** | 3 | 27.2 | Equivalent functions | 2 | 18.1 |
| **No solution** | 0 | | No solution | 0 | 0 |
| **Total** | 11 | 100 | Total | 11 | 100 |

Both pre-service and in-service teachers gave satisfactory answers to this question. Nonetheless, some pre-service teachers gave reasons that are not mathematically correct, like the one shown in answer B. Figure 4 shows two prospective teachers' answers.

**Figure 4**

*Answers associated to configurations "Uniqueness" and "Equivalent functions"*

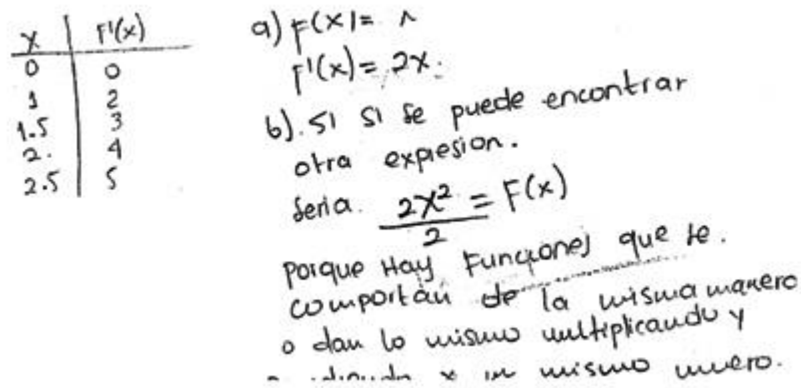
Answer A: Configuration 'Uniqueness'




b) *$F(x) = x^2 + C$, because for any constant C its derivative is zero.*

Answer B: Configuration ‘Equivalent functions’

a)




b) *Yes, another expression can be found, it would be f(x) = $2x^2/2$, because there are functions whose behaviour are the same multiplying or dividing by the same number.*

Figure 5 shows a solution given by two in-service teachers.

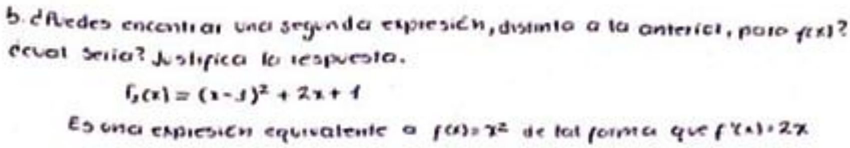
**Figure 5**

*Cognitive configurations: derivative as slope of straight-line tangent*

Answer A: Configuration “Verbal”

b) *Si, $f(x) = x^2 + c$; it can be found an entire family of functions, because c is a constant, and its derivative is zero.*

Answer B: Configuration "Examples"


b. ¿Puedes encontrar una segunda expresión, distinta a la anterior, para f(x)? ¿Cuál sería? Justifica la respuesta.

$f_1(x) = (x-1)^2 + 2x + 1$

Es una expresión equivalente a $f(x)=x^2$ de tal forma que $f'(x)=2x$

b) *$f(x) = (x-1)^2 +2x+1$, it is an equivalent expression to $f(x) = x^2$ so that $f'(x) = 2x$.*

The configurations labelled as 'Technical' are those in which theorems were used to derive, answer, and justify solutions in sections (a) and b). The term 'Advanced' is assigned to those solutions in which more advanced concepts were used, such as the integral or the fundamental calculus theorem. Thus, we see that in Item a) of the task, 27.2 % of prospective teachers used a graphic-technical configuration, i.e., the derivative function is calculated from the graphical interpretation of data given in the table and by using the derivation rules, the function f(x) is found. 27.2 % gave an answer that is associated with a numerical-advanced type configuration, i.e., from data in a given table, they found a pattern that allowed them obtaining the correspondence rule that defines the derivative function and, by using concepts such as the integral, they found the expression for f(x). The results obtained in Task 3 support the need for improving the prospective teachers' extended knowledge, enabling them to solve tasks such as the one set out.

**Results for Task 4: The derivative of a constant function**

For Task 4, we considered correct those answers in which respondents used graphic representations and verbal descriptions to justify the proposition 'the derivative of a constant function is always zero,' partially correct, the answers that provided valid graphs, but which verbal descriptions did not justify the initial proposition, and incorrect those answers in which students did not provide graphs nor accurate verbal descriptions to justify their proposition. Table 11 and Table 12 show the results for the degree of correctness of Task 4 for pre-service and in-service teachers, respectively.

**Table 11**

*Degree of correction of Task 4: Pre-service teachers*

| Level of accuracy | Section a) | | Section b) | |
|---|---|---|---|---|
| | Frequency | % | Frequency | % |
| **Correct** | 8 | 36.3 | 3 | 13.6 |
| **Partially correct** | 7 | 31.8 | 4 | 18.1 |
| **Incorrect** | 5 | 22.7 | 8 | 36.3 |
| **No answer** | 2 | 9.0 | 7 | 31.8 |
| **Total** | 22 | 100 | 22 | 100 |

**Table 12**

*Degree of correction of Task 4: In-service teachers*

| Level of accuracy | Section a) | | Section b) | |
|---|---|---|---|---|
| | Frequency | % | Frequency | % |
| **Correct** | 11 | 100 | 11 | 100 |
| **Partially correct** | 0 | 0 | 0 | 0 |
| **Incorrect** | 0 | 0 | 0 | 0 |
| **No answer** | 0 | 0 | 0 | 0 |
| **Total** | 11 | 100 | 11 | 100 |

As shown in Table 11, only 36.3% of prospective teachers correctly solved Item a) of Task 4. The above results suggest that more than half of the prospective teachers had difficulties solving that Item. 13.6% of teachers answered Item b) correctly, which shows that more than half of the prospective teachers have difficulties demonstrating, by the formal definition of the derivative, the proposition ‘the derivative of a constant function is always zero.’ The latter shows that more than 50% of prospective teachers demonstrate inadequacy regarding the epistemic facet of the DMK required to solve the task.

Regarding the configuration, Table 13 shows the results of Task 4 for pre-service teachers.

**Table 13**

*Cognitive configuration of item a) for Task 4: Pre-service teachers*

| Cognitive configuration | Task 4: item a) | |
|---|---|---|
| | Frequency | % |
| **Analytical – extensive** | 8 | 36.3 |
| **Analytical – intensive** | 6 | 27.2 |
| **Tangents drawing** | 2 | 9.0 |
| **Use of particular situations of variation** | 1 | 4.5 |
| **Limit of the average rates of variation** | 3 | 13.6 |
| **No solution** | 2 | 9.0 |
| **Total** | 22 | 100 |

Table 13 indicates that analytical-extensive, analytical-intensive settings, and tangents drawing account for 72.5 % of prospective teachers' solutions in which the derivative is interpreted as the slope of the straight-line tangent. Analytical-extensive refers to, for both pre-service and in-service teachers, the centrepiece of cognitive configurations associated with resolutions of Task 4 is the statement 'the derivative of a constant function is always zero'. Thus, in the 'analytical-extensive' cognitive configuration, the arguments given are special cases of calculating the slopes of some horizontal lines. In the 'analytical-intensive' configuration, justifications are based on the calculations of the slope of a generic constant function. For the configuration 'tangent drawing', justifications are given based on the drawing of straight lines tangent to the constant function. Finally, 'drawing of tangents,' 'justifications are based on straight lines tangents' drawings to the constant function. In the configuration 'use of particular situations of variation,' justifications are based on specific cases of variation, specifically velocity. Finally, in the configuration 'limit of average rates of variation,' the arguments are supported on the notion of instantaneous rate of variation, without considering specific cases such as velocity.

Figure 6 shows examples provided by pre-service teachers and illustrates the first three types of configurations on which the derivative, a slope of the straight-line tangent, is considered. Analytical-extensive refers to solutions based on examples, while analytical-intensive refers to solutions based on demonstrations. Tangents drawing refers to the drawing on tangents to explain de solution. Some solutions refer to examples of variation to justify the solutions. It is interesting noting that the pre-service teachers gave quite satisfactory graphic solutions to Task 4. Incidentally, it is a quite common task in calculus textbooks; nonetheless, as Figure 14 and Table 17 show, this is not the case for Task 6, where the graphic setting has been proved difficult for pre-service teachers.

**Figure 6**

*Cognitive configurations: Derivative as slope of straight-line tangent*

Answer A: "Analytical-Extensive" cognitive configuration

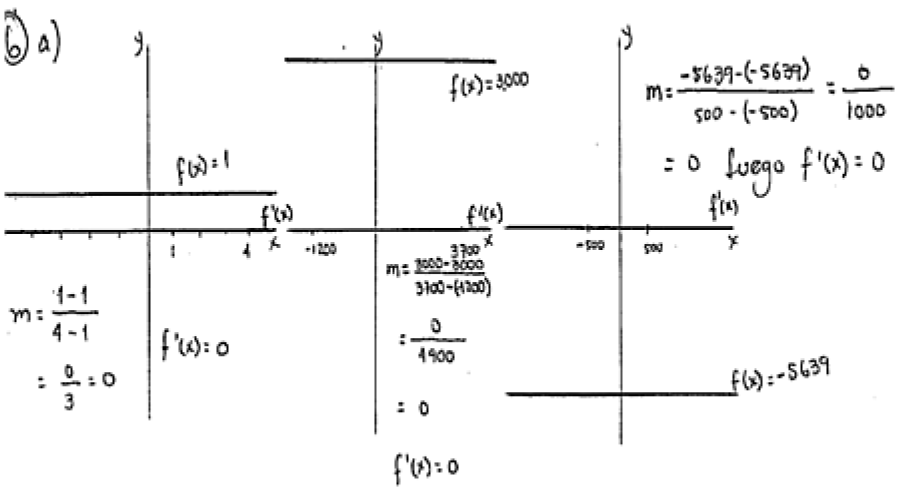
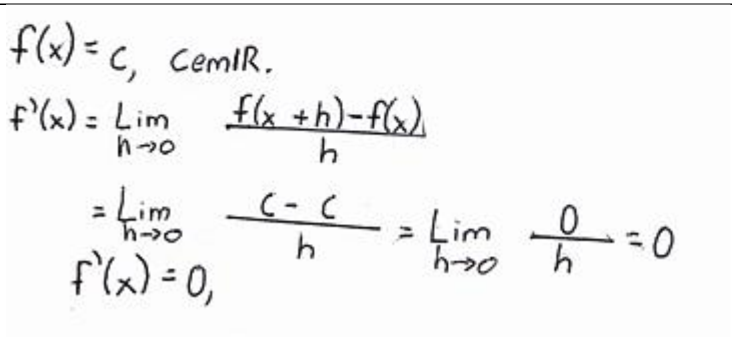
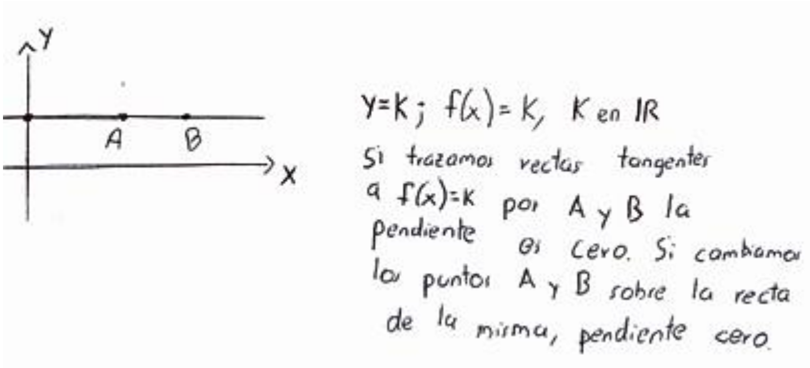



**Figure 6 (cont'd)**

Answer B: "Analytical-Intensive" cognitive configuration




Answer C: “Tangents-drawing” cognitive configuration




If we draw tangent straight lines to f(x) = k through A and B the slope is zero. If we change the points A and B over same line, the slope is zero.

For in-service teachers, Table 14 shows the results of Task 4 regarding the type of configuration.

Table 14 shows that considering the analytical-extensive, analytical-intensive settings and tangents drawing, 54.4% of in-service teachers provided a solution in which the derivative was interpreted as the slope of the straight-line tangent. The centrepiece of cognitive configurations associated with resolutions is the arguments or justifications to the proposition ‘the derivative of a constant function is always zero’.

**Table 14**

*Cognitive configuration for item a) Task 4: In-Service teachers*

| Cognitive configuration | Task 4: item a) | |
|---|---|---|
| | Frequency | % |
| **Analytical – extensive** | 3 | 27.2 |
| **Analytical – intensive** | 3 | 27.2 |
| **Tangents drawing** | 1 | 9.0 |
| **Use of particular situations of variation** | 1 | 9.0 |
| **Limit of the average rates of variation** | 3 | 27.2 |
| **No solution** | 0 | 0 |
| **Total** | 11 | 100 |

Thus, in the 'analytical-extensive' cognitive configuration, the arguments given are special cases of calculating the slopes of some horizontal lines. In the 'analytical-intensive' configuration, justifications are done by calculating the slope of a generic constant function. For the configuration 'tangent drawing,' justifications are given from the drawing of straight lines tangent to the constant function. Lastly, 'drawing of tangents,' the justifications are based on drawings of straight lines tangents to the constant function. In the configuration 'use of particular situations of variation,' justifications are based on specific cases of variation, specifically velocity. Finally, in the configuration 'limit of average rates of variation,' the arguments are supported on the notion of instantaneous rate of variation, without considering specific cases such as velocity.

Given the relation between Task 4 and the type of knowledge that it evaluates, we infer that prospective teachers lack knowledge: the epistemic facet of the DMK (use of different representations, use of different meanings of the derivative, solving the problem by various procedures, give several valid arguments to explain these procedures, etc.), and the required common content knowledge.

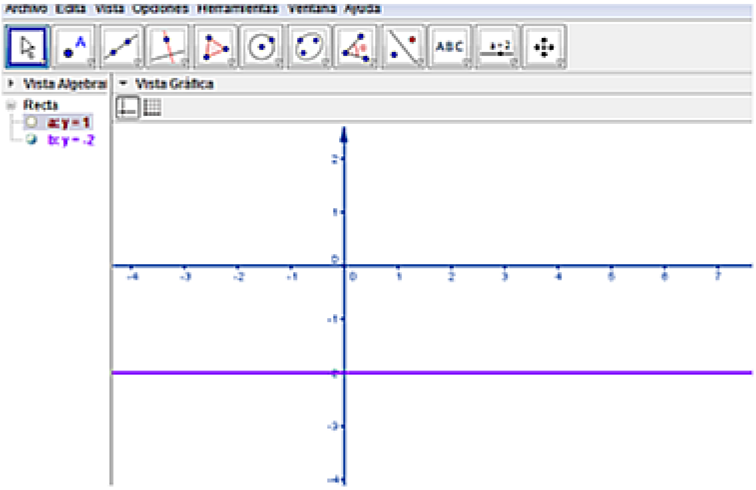
Figure 7 shows examples that illustrate three types of configurations corresponding to in-service teachers.

## Figure 7

*"Analytical-Extensive" and "Analytical-Intensive" cognitive configurations*

Answer A: "Analytical-Extensive" cognitive configuration

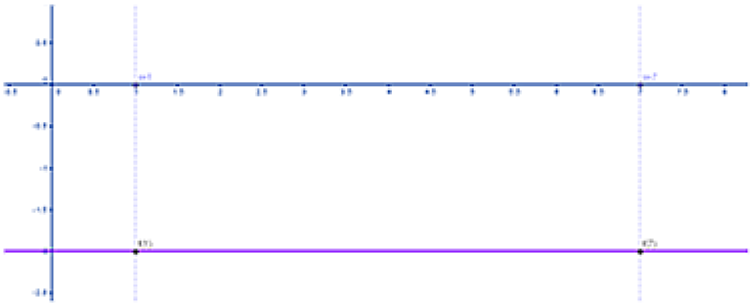
The following is the graphic representation for f(x) = -2, where f is the constant function.




The derivative of this function is the change of rate that take images in regard to the values taken in the domain.

## Figure 7 (cont'd)

Answer A (cont'd):



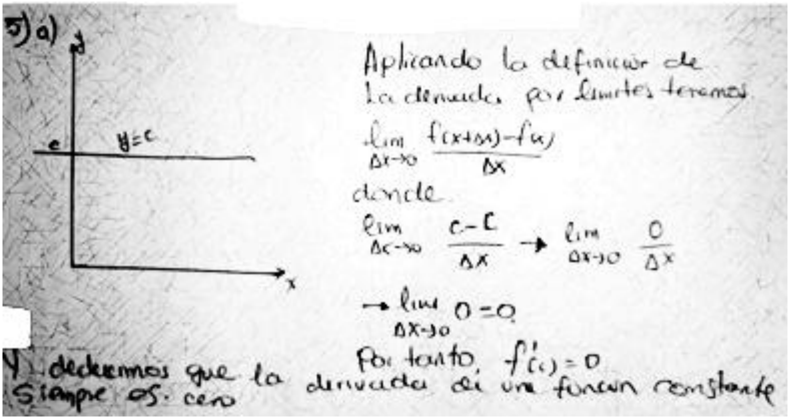
$$Derivada = \frac{Cambio\ en\ las\ imagenes\ repectivas\ al\ dominio}{Cambio\ en\ los\ valores\ del\ dominio}$$

$$\frac{f(7) - f(1)}{7 - 1} = \frac{-2 - (-2)}{6} = \frac{0}{6} = 0$$

Como se identificó, una función constante tiene como valor de sus imágenes el mismo para todos los valores que tome su dominio, por lo tanto el cambio entre sus imágenes siempre será 0.

Derivative = Change in the corresponding domain images /Change in the domain values as it was identified, a constant function has the same values for all domain values, thus the change among images is always zero.

Answer B: "Analytical-Intensive" cognitive configuration



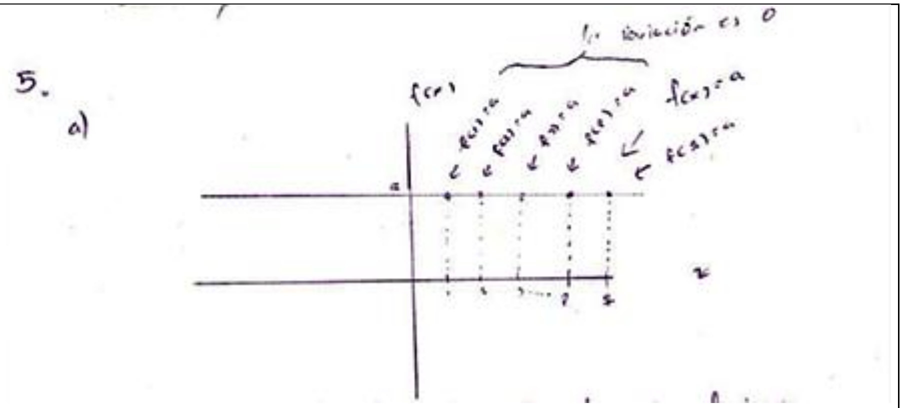

Applying the definition of the derivative by limits, we obtain… thus, f'(x) = 0 and we deduce that the derivative of a constant function is always zero.

Not a single in-service teacher proposed the "Tangents-drawing" cognitive configuration; nonetheless, one teacher proposed the graphic explanation shown in Figure 8. The latter resort to the fundamental idea of change that seems to be close to his students. According to Hatisaru and Erbas (2017), the many meanings and representations of the function, concepts make it difficult for teachers to deal with them all and establish relations among meanings, and it comes even more complicated when the derivative is included, the more the meanings, the more complex for teachers to deal with the function topics.

Panero, Arzarello, and Sabena (2016), report that "…the global perspective on the derivative function might be enhanced, but it is necessary to establish carefully the relationships between the graph and the semiotic resources (i.e., symbols and speech) through which the derivative function has been introduced, strengthening the meaning of the used signs and variables" (p.282).

**Figure 8**.

An alternative explanation by in-service teacher.



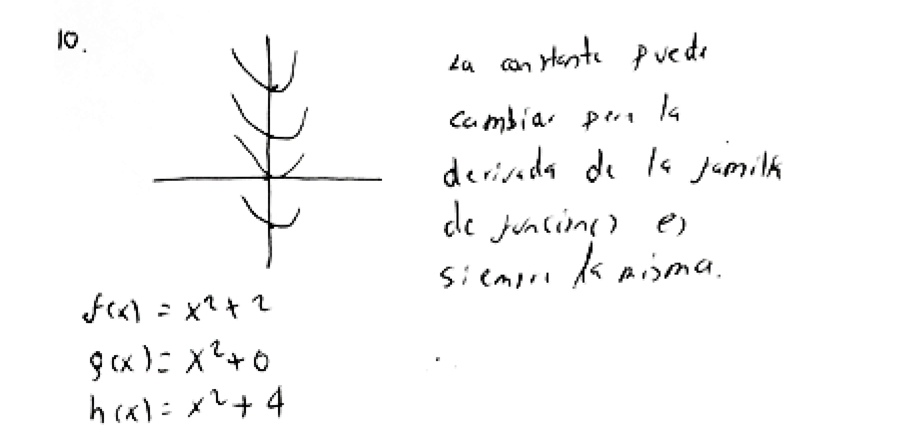

The variation is zero.

Another solution proposal by an in-service teacher is shown in Figure 9.

Given the relation between Task 4 and the type of knowledge that it evaluates, we can say that in-service teachers have both, knowledge of the epistemic facet (use of different representations, use of different meanings of the derivative, solving the problem by various procedures, give several valid arguments to explain these procedures, etc.) and common content knowledge required to teach the derivative efficiently.

**Figure 9**

*A graphic explanation*



The constant can change but the derivative of the family of functions is always the same.

### Results for Task 5: Zeros of the derivative of a function

Table 15 shows the results obtained for Task 5 concerning the degree of the correctness. Since the task explores whether respondents establish connections among different derivative' meanings, for Section b) of Task 5, those answers which proposed associations among the different meanings of the derivative and whose justifications were valid were considered correct. Those answers in which connections are not made between meanings of the derivative as the slope of the straight-line tangent and instantaneous rate of change were considered incorrect. Those answers that establish connections among different meanings of the derivative but whose justifications are not entirely valid, were considered partially correct.

As we can see in Table 15, 36.3% of prospective teachers had no trouble responding Item a) of the task. However, only 18.3% were able to answer Section b) correctly). The results obtained for Task 5 and previous tasks point out a disconnection between meanings of the derivative that they know and those used in mathematical practices concerning the derivative. Figure 10 shows Sandra's solution. She provides different meanings of the derivative for Task 1, including "…as the slope of a straight line…" and "... the rate of change...,” but does not establish a link among those meanings in Task 5. Akkoç, Bingolbali, and Ozmantar (2008) report that teachers could not relate the notion of rate of change to a graphical or numerical meaning of the derivative.

**Table 15**

*Degree of correction of Task 5: Pre-service teachers*

| **Level of accuracy** | **Section a)** | | **Section b)** | |
|---|---|---|---|---|
| | **Frequency** | **%** | **Frequency** | **%** |
| **Correct** | 8 | 36.3 | 4 | 18.3 |
| **Partially correct** | 7 | 31.8 | 9 | 40.9 |
| **Incorrect** | 4 | 18.1 | 3 | 13.6 |
| **No answer** | 3 | 13.6 | 6 | 27.2 |
| **Total** | 22 | 100 | 22 | 100 |

This is the case for pre-service teachers in our study. Both items are related to finding where the derivative of a polynomial function is zero, nonetheless, some pre-service teachers seem to ignore that the points where the tangent is horizontal are the same points where the instantaneous rate of change is zero. Teachers do not relate different meanings that are closed related. Regarding approaches for teaching derivative, Rosnawati, Wijaya, and Tuharto (2020) say that teachers used a graphical approach because they emphasised the conceptual understanding of derivative, including gradient, rate of change, border, and symbolical understanding.

Table 16 shows frequencies and percentages of degree of correctness of in-service teachers’ solutions for Task 5.

As it can be seen in Table 16, 81.8% of prospective teachers had no trouble responding Item a) of the task. Almost all in-service teachers (90.9%) solved Section b) correctly.

**Table 16**

*Degree of correction of Task 5: In-service teachers*

| **Level of accuracy** | **Section a)** | | **Section b)** | |
|---|---|---|---|---|
| | **Frequency** | **%** | **Frequency** | **%** |

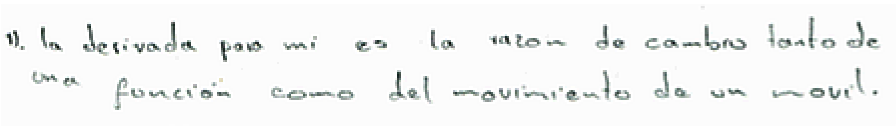
| | | | | |
|---|---|---|---|---|
| **Correct** | 9 | 81.8 | 10 | 90.9 |
| **Partially correct** | 2 | 18.1 | 1 | 9.0 |
| **Incorrect** | 0 | 0 | 0 | 0 |
| **No answer** | 0 | 0 | 0 | 0 |
| **Total** | 11 | 100 | 11 | 100 |

Figures 10, 11, and 12 show some solutions to Task 5. It is interesting to notice that symbolic solutions seem to derive from verbal solutions. The more general the verbal solution, the more imprecise the symbolic solution.

**Figure 10**

*Sandra's answer to Task 1 and Task 5*

Answer to task 1: What is the derivative of a function?




The derivative is the rate of change of both of a function and the movement of an object.

Answer to Task 5

a) Find the points where the graphic of the function has a horizontal tangent.

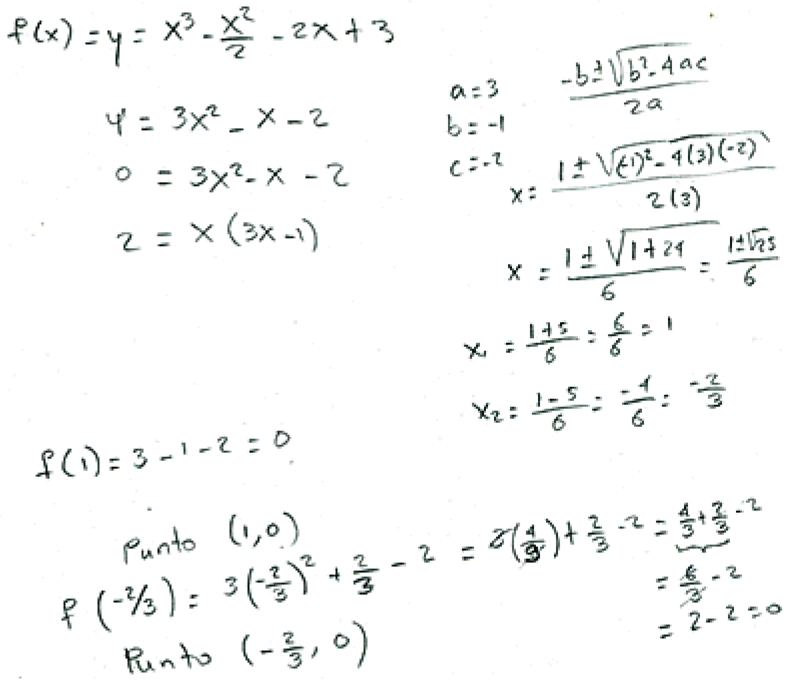
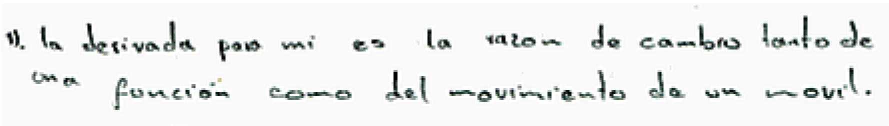
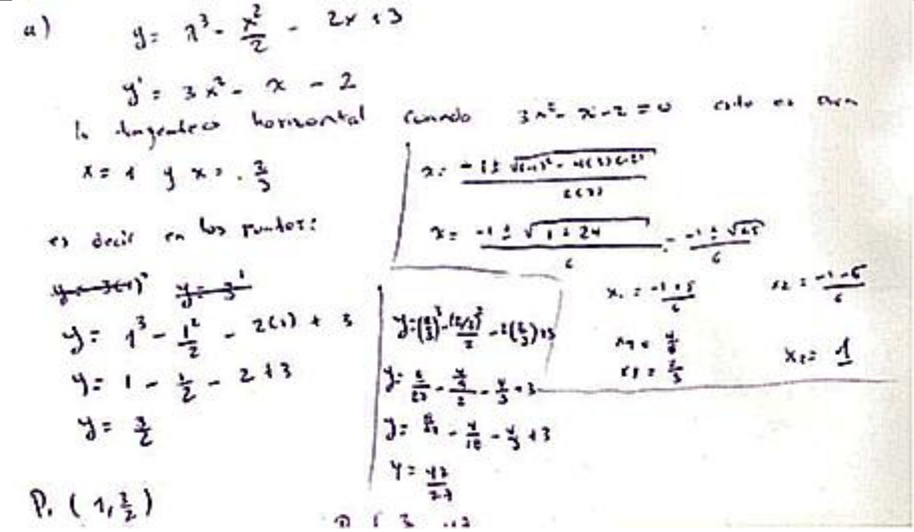
| |
|---|
| <br> |
| b) What are the points where the instantaneous rate of change of *y* in regard to *x* is zero? |
| <br> No answer. |

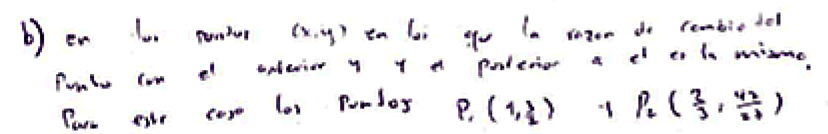
**Figure 11**

*Sam's solutions to Task 1 and Task 5*

| Answer to Task 1: What is the derivative of a function? |
|---|
| <br><br>The derivative, to me, is a rate of change of a function and the movement of an object as well. |
| Answer to Task 5<br>a) Find the points where the graphic of the function has a horizontal tangent. |



b) What are the points where the instantaneous rate of change of y in regard to x is zero?

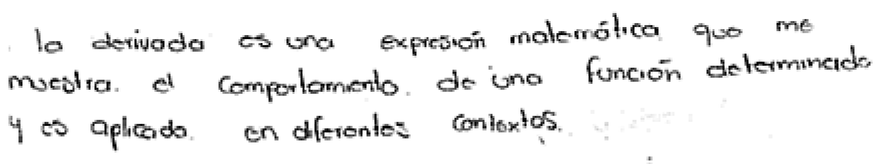


In the points (x,y), where the rate of change with the previous point and the posterior to it is the same, for this case the points P1 (1, 3/2) and P2 (2/3, 47/27).

**Figure 12**

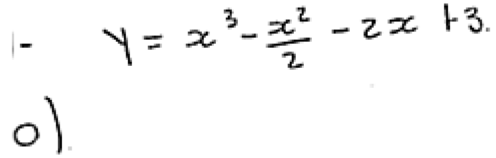
*Carlos' solutions to Task 1 and Task 5*

Answer to Task 1: What is the meaning of the derivative?




The derivative is a mathematical expression that shows the function behaviour, and it is applied in manifold contexts.

*Answer to Task 5*

| |
|---|
|  |
| Do not respond to neither question |

According to De Almeida and Da Silva (2018), “the process of conceptualisation and meaning making for mathematical objects (i.e., concepts) can be seen as a recursive process mediated by a diversity of mathematical signs” (p. 883). The fruitful and intrinsic relationship between sign vehicle, object, and interpretant is an instrument for describing learning and teaching mathematics, a way of interpreting classroom communication (De Almeida & Da Silva, 2018). Informants can certainly solve mathematics tasks, but when it comes to establishing or identifying relationships among mathematics meanings, the results show a lack of such knowledge. It seems that the knowledge is attached to the meanings of the representations used to present either the question or the solution.

Analysing teachers’ discourses, Park (2015) highlights “specific disconnections between, on the one hand, mathematical concepts and, on the other hand, the words, symbols, graphs, and gestures used to communicate them” (p. 249). Orton (1983) provides one of the earliest descriptions of student difficulties with derivatives; while the students he studied were generally proficient at computing derivatives, he found significant misunderstandings of the derivative as a rate of change, results that coincide with our findings. A naïve stance toward in-service teacher’s solutions could question the lack of connectedness among mathematics meanings in pre-service teachers’ solutions, nonetheless, this lack is the result of both teaching practices in university mathematics content knowledge courses, and the complexity of mathematics knowledge.

So, as important as discussing meanings are the connections among meanings. In what follows, different answers to Task 5 are presented.

Figure 11 shows how Sam begins to resolve Section a) of Task 5, in the same way that she proceeds later in Section b). However, when she realises what is asked in Item b) of Task 5, she writes as a response, “In the points (x, y), where the rate of change with the previous point and the posterior to it is the same…”. Subsequently, she answers Section b) correctly, finding the points at which the rate of change of x concerning y is zero. Nonetheless, her explanation is a little strange. More notable is the case of Carlos (Figure 12), who provides meanings for the derivative such as: “The derivative is a mathematical expression that shows the function behaviour, and it is applied in manifold contexts”. The procedural interpretation of such definition leads him to not answering the question. Gökçek and Açıkyıldız (2016) reported that the pre-service teachers were more successful in interpreting the derivate concept in algebraic form than graphical and table representations.

As shown in the previous figures (Figures 10, 11, and 12), prospective teachers have difficulties establishing connections between the two meanings of the derivative. As it is shown in Table 11, 86.2 % of prospective teachers failed to make an association between the meanings of the derivative as follows: “the rate of change of y with respect to x is zero at those points where the straight-line tangent to the function is horizontal”.

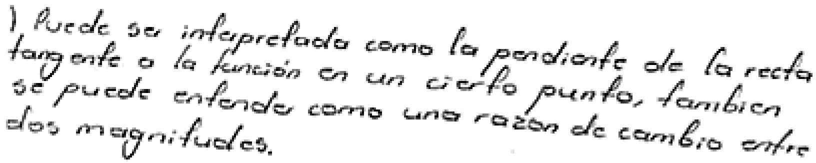
In contrast, the answers provided by in-service teachers show that such a connection is well known. Figure 13 shows an example of an in-service teacher’ solution to this task.

Hill et al. (2008) reported that “teachers with stronger Mathematics Knowledge for Teaching (MKT) (including common and specialised mathematics content knowledge) made fewer mathematical errors, responded more appropriately to students, and chose examples that helped students construct meaning” (p. 284). The epistemic facet of the derivative is not enough to teach. Still, it seems obvious that without such knowledge, the teacher could not design a class that complies with the specialised content knowledge characteristics.

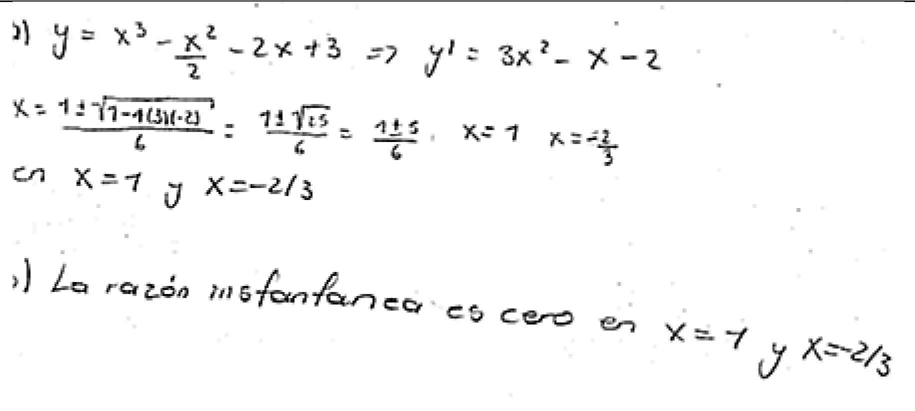
**Figure 13**

*Camilo’s answers to Task 1 and Task 5*

Answer to Task 1 - What is the derivative of a function?


1) Puede se interpretada como la pendiente de la recta tangente a la función en un cierto punto, tambien se puede entende como una razon de cambio entre dos magnitudes.

It can be interpreted as the slope of a tangent line to a function in a given point; it also can be understood as the rate of change between two magnitudes.




The rate of instantaneous change is zero in x=1 y x= -2/3

**Results for Task 6: Instantaneous rates of variation**

Table 17 displays the results for the degree of correction of Task 6. As presented, 54.5 % of the prospective teachers had difficulties solving the task. However, only 13.6 % correctly responded to Section a) of Task 6. Figure 14 shows a prototypical example of partially correct answers given by pre-service teachers.

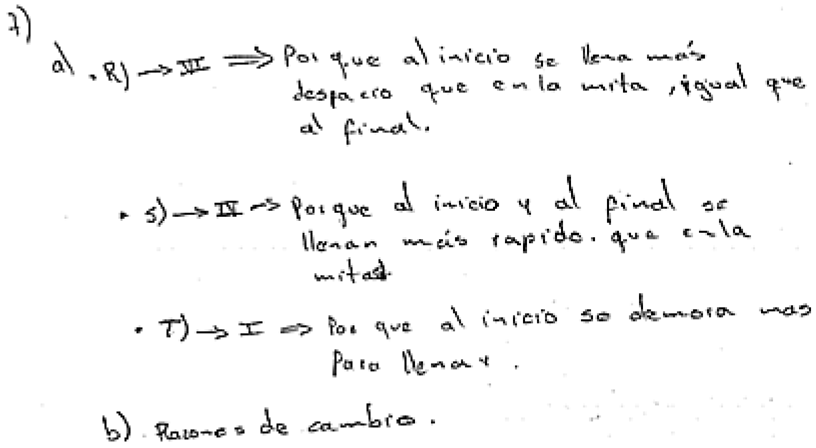
**Table 17**

*Degree of correction of Task 6: Pre-service teachers*

| Level of accuracy | Task 6: item a) | |
|---|---|---|
| | Frequency | % |
| **Correct** | 3 | 13.6 |
| **Partially correct** | 9 | 40.9 |
| **Incorrect** | 6 | 27.2 |
| **No answer** | 4 | 18.1 |
| **Total** | 22 | 100 |

**Figure 14***nswer partially correct for section a) of Task 6: pre-service teachers*

Answer 1.


7)
a) • R) → III ⟹ Por que al inicio se llena más despacio que en la mita, igual que al final.
• S) → II → Porque al inicio y al final se llenan más rapido. que en la mitad
• T) → I ⟹ Por que al inicio se demora mas para llenar.
b) Razones de cambio.

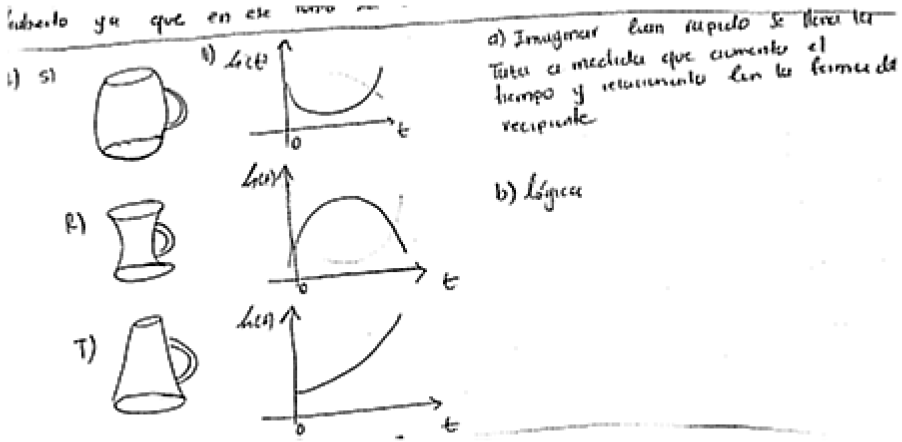
R. Because at the beginning it fills slower that in the middle, the same at the end

S. Because at beginning and at the end the vessels fill faster that in the middle

T. Because at the beginning it takes more [time] to fill

**Figure 14 (cont'd)**

Answer 2.




d) To imagine how fast the vessel is fill while the time increases, and to relate it to the vessel shape.

b) Logic

Table 18 displays the results for the degree of correction of Task 6 of in-service teachers. We can observe that 72.7 % of teachers had no difficulties solving the task. Just one teacher did not complete it. It is interesting noticing that in-service teachers have the meanings required to solve the task.

Figure 15 shows a prototypical example of partially correct answers given by the in-service teachers. Teachers' epistemic knowledge is key for teaching because "the conceptualisation process intended to be carried out by signs is co-constructed by those who teach or by the material presenting what is to be conceptualised by the student…" (De Almeida & Da Silva, 2018, p. 883).

As in Çetin's (2009) study, 27.2% of prospective teachers did not establish a relation between the functions h(t), represented by the cups, and graphs of the functions h'(t). One of the possible causes is that pre-service teachers are not accustomed to identifying mathematics concepts in everyday situations, which hindered the passage from the iconic representation of the function (the drawing of the cups) to the graphical representation of the derivative function, without using the function graphic representation. The importance task representation is unconventional, that, according to Zazkis

(2008), it is a vehicle towards constructing a “richer or more abstract schema” (p.154). As reported in various studies (Haciomeroglu et al., 2010; Mhlolo, 2012; Moon et al., 2013), students and teachers find it difficult to make mathematical connections between different representations.

**Table 18**

*Degree of correction of Task 6: In-service teachers*

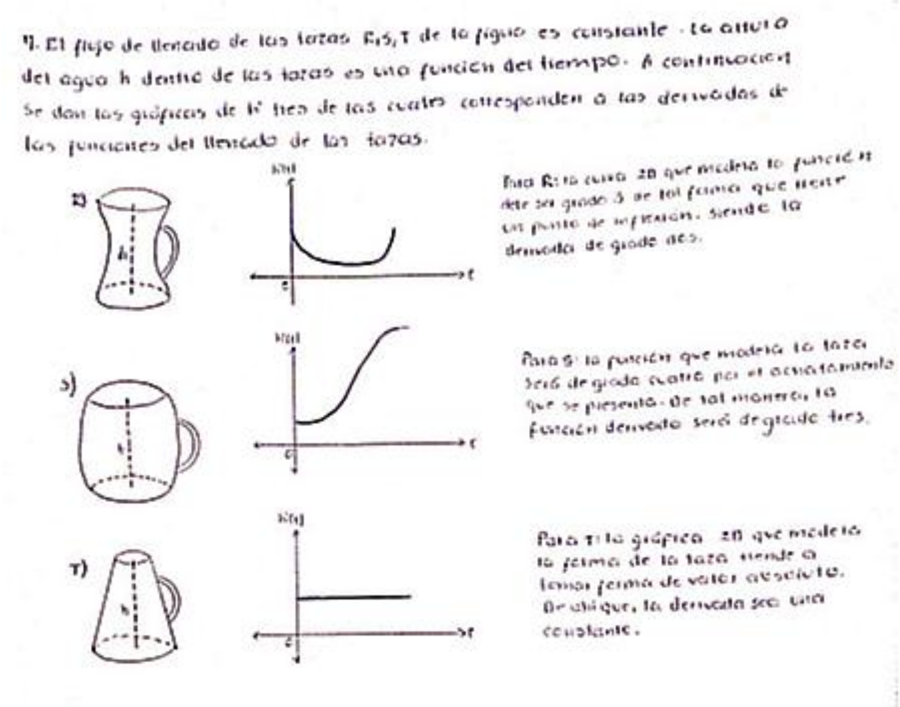
| **Level of accuracy** | **Task 6 item a)** | |
|---|---|---|
| | **Frequency** | **%** |
| **Correct** | 8 | 72.7 |
| **Partially correct** | 2 | 18.1 |
| **Incorrect** | 1 | 9.0 |
| **No answer** | 0 | 0 |
| **Total** | 0 | 0 |

Another difficulty refers to the relationship among point, global, and local properties. Maschietto (2008), Rogalski (2008), and Vandebrouck (2011) distinction between pointwise, global, and local properties on a given real function f of one real variable is very complex.

Regarding Section b) of Task 6, all prospective teachers who responded provided ‘lists’ of concepts such as derivative, function, modelling, concavity, growth, and decreasing of functions, etc., which stress that their epistemic facet of the DMK should be encouraged. In-service teachers gave an explanation based on rate of change, Figure 15, while the pre-service teachers affirm that the assignment is done using ‘logic’. In Task 4, 36.3% and 13.6 % of pre-service teachers gave correct solutions to Task a) and Task b), respectively, in contrast to 4.5 % of correct solutions given to Task 6.

**Figure 15**

*Answer partially correct for section a) of Task 6: in-service teachers*




R) The curve 2D that models the function should be of grade 3, so that it has an inflexion point, and the derivative is of grade 2.

S) The function 2D that models the rate will be of grade four due to its shape; in such a way, the derivative function will be of grade three.

T) The graphic 2D that models the shape of the vessel tends to a shape of absolute value. Thus, the derivative must be a constant.

**Figure 15 (cont'd)**

Para R le correspondería la gráfica II
para S le correspondería la gráfica V
para T le correspondería la gráfica I

| |
|---|
| For R corresponds to graphic II<br>For S corresponds to graphic V<br>For T corresponds to graphic I |

It seems that the relations among real-life containers and derivative graphic representations proved to be difficult; incidentally, these types of tasks are not common in calculus textbooks. Habre and Abboud (2006) report that only one of ten students thought of the function as a graph, and only three others associated a graphical representation with a function that coincides with our results. Studies (Dreyfus & Eisenberg, 1990; Tall, 1991) have consistently shown that students' understandings are typically algebraic and not visual; visual information is more difficult for students to learn and is considered less mathematical.

**Results for Task 7: Instantaneous velocity**

For Task 7, the solutions were considered correct if they included the ball's speed. The correct answers are related to the setting by bilateral approximation. Partially correct answers related to left or right approximation were those in which procedures and justifications that are not entirely erroneous were used, but they are either not valid enough to find the speed. As incorrect answers, we considered those in which the speed of the ball to t = 0.4 is not found, because the procedures and justifications were not valid. Incorrect answers are related to the types of numeric pattern and the use of the physical ratio v = d/t configurations. Table 19 shows the results for Task 7.

**Table 19**

*Degree of correction of task 7: Pre-service teachers*

| Level of accuracy | Task 7 | |
|---|---|---|
| | Frequency | % |
| **Correct** | 1 | 4.5 |
| **Partially correct** | 7 | 31.8 |
| **Incorrect** | 13 | 59 |
| **No answer** | 1 | 4.5 |
| **Total** | 22 | 100 |

From Table 19, we can conclude that only one prospective teacher could solve the task correctly, and five others give a partially correct answer. The latter suggests that at least 63.7% of the prospective teachers lack the required extended mathematical knowledge to solve the task. This task is related to Task 5 b) (instantaneous rate of change); in both cases, the percentage of correct solutions are relatively low for the pre-service teachers. It seems that not only the meaning of derivative as instantaneous rate of changing is lacking in pre-service teachers' knowledge, but also the task's representation influences the percentage of correct answers.

According to Muhundan (2005), if students learned how to solve a rate of change problem on a function by a numerical approach with the help of a graphic calculators, they "will be ready to solve a rate of change problem on any type of function because the procedures and the amount of work needed to do the problem are the same. It seems that the correctness of students' solutions is related to previous learning experiences based on algebraic techniques, manipulations, or formulas to solve a task. Ozmantar, Akkoç, Bingolbali, Demir, and Ergene (2010) report in their research that students managed to give examples of multiple representations of the derivative in graphical, tabular (numerical), and algebraic tasks after they took a course with emphasis on the use of computers to explore the multiple representations of the derivative. The degree of correctness results could be interpreted not only in relation to common content knowledge but also in relation to school teaching tradition.

Finally, Table 20 presents the results concerning the variable 'cognitive configuration' for Task 7 for pre-service teachers.

For in-service teachers, Table 21 presents the results concerning the percentages for the degree of correctness of Task 7. From Table 21, we can see that almost all in-service teachers (72.7%) solved the task correctly, and only two give a partially correct answer.

Finally, Table 22 presents the results concerning the variable 'cognitive configuration' for Task 7, used for the in-service teachers.

**Table 20**

*Cognitive configurations for Task 7: Pre-service teachers*

| Type of cognitive configuration | Task 7 | |
|---|---|---|
| | Frequency | % |
| Numerical Pattern | 6 | 27.2 |
| Use of physical ratio $v = d/t$ | 10 | 45.4 |
| Left or right approximation | 2 | 9.0 |
| Bilateral approximation | 4 | 18.1 |
| No evidence of solution | 0 | 0 |
| Total | 22 | 100 |

**Table 21**

*Degree of correction of Task 7: In-service teachers*

| Level of accuracy | Task 7 | |
|---|---|---|
| | Frequency | % |
| Correct | 8 | 72.7 |
| Partially correct | 2 | 18.1 |
| Incorrect | 1 | 9 |
| No answer | 0 | 0 |
| Total | 11 | 100 |

According to Kunter, Klusmann, Baumert, Richter, Voss, and Hachfeld (2013), teachers who scored high on pedagogical content knowledge provided more cognitively activating instruction and better learning support to students, “with the former showing positive effects on student achievement and the latter on student motivation” (p. 815). This finding confirms earlier research (Baumert et al., 2010; Hill et al., 2008), underscoring the importance of profession-specific knowledge for fostering students’ learning processes.

**Table 22**

*Cognitive configurations for Task 7: In-service teachers*

| Type of cognitive configuration | Task 7 | |
|---|---|---|
| | Frequency | % |
| **Numerical Pattern** | 2 | 18.1 |
| **Use of physical ratio $v = d/t$** | 7 | 63.6 |
| **Left or right approximation** | 1 | 9.0 |
| **Bilateral approximation** | 1 | 9.0 |
| **No evidence of solution** | 0 | 0 |
| **Total** | 11 | 100 |

Interestingly, in our study, both pre-service and in-service teachers expressed difficulties solving derivative problems in numerical terms. The results of their solutions to Tasks 3 and Task 7 back our statement. The same can be said when comparing Task 5 b) and Task 7, in-service teachers' percentage of correct solutions are higher than pre-service teachers', nonetheless Task 7 proved even more difficult for in-service teachers - 72.7% gave correct solutions. According to Habre and Abboud (2006), "the numerical representation of a function was not on any student's mind, perhaps because, in the course, less emphasis was placed on the numerical representation of the function" (p. 62).

From Tables 19 and 22, and the analysis performed for each type of cognitive configuration, we can conclude that 45.4 % (Table 20) of prospective teachers use the ratio v = d/t to solve the task. Other studies (Inglada & Font, 2003; Badillo, Azcárate, & Font, 2005; Badillo, Azcárate, & Font, 2011) reported the same results about the difficulty to establish the difference between the derivative function and the derivative. It seems there is an apparent separation between the interpretation of the derivative at a point and the derivative function, which can mislead respondents to wrong solutions when answering the question: What is the speed of the ball at time t = 0.4s? by calculating the average velocity. It seems that teachers do not relate different meanings to the derivative. Students seem to ignore that the relation v = d/t represents the average speed of the ball for two different time values. This average speed, in turn, is related to the slope of some straight-line secant to the displacement function, which does not correspond to an interpretation for the derivative. Based on in-service teachers' task's solutions to Item 6 and Item 7, we can deduce that they are not used to solving derivative tasks that resource to graphic and numerical strategies. In her study, Lam (2009) reported that subjects had difficulties not only finding derivatives of functions that do not have an easily available formula for differentiation, but also interpreting graphically the first and second derivatives in relation to the graphs of the functions they represent. Orton (1983) found significant misunderstandings in the graphical representation of the derivative.

**Results for Task 8: Derivative of the exponential**

For Task 8, we do not consider types of cognitive configurations, just the types of assessment offered by pre-service and in-service teachers: the derivative is correct, the derivative is incorrect, and it is incorrect with a justification. Table 23 shows frequencies and percentages of the type of assessment of in-service teachers' solutions to Task 8.

**Table 23**

*Cognitive configuration for Task 8: Pre-service and in-service teachers*

| Type of assessment | Pre-service | | In-service | |
|---|---|---|---|---|
| | **Frequency** | **%** | **Frequency** | **%** |
| **The derivative is correct** | 15 | 68.1 | 0 | 0 |
| **The derivative is incorrect** | 0 | 0 | 4 | 36.3 |
| **It is incorrect with a justification** | 7 | 31.8 | 7 | 63.6 |
| **Total** | 22 | 100 | 11 | 100 |

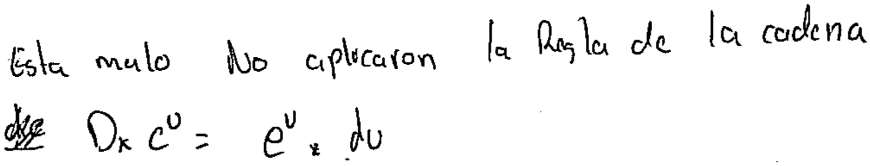
Figure 16 shows an example of every type of solution. Solutions A and B by pre-service teachers, and C by in-service teacher.

Neither the pre-service nor the in-service teachers gave an alternative explanation to the student, as was asked in the item. The results of the study conducted by Saltan and Arslan (2017) showed that both pre-service and in-service teachers need to improve their knowledge and skills to meet the requirements to teach. In-service teachers may have more mathematics knowledge for teaching, but it seems that teachers' epistemic facet of the didactic-mathematical knowledge alone does not ensure effective teaching performance (Kahan, Cooper, & Bethea, 2003).

**Figure 16**

*Examples of teachers' solutions*

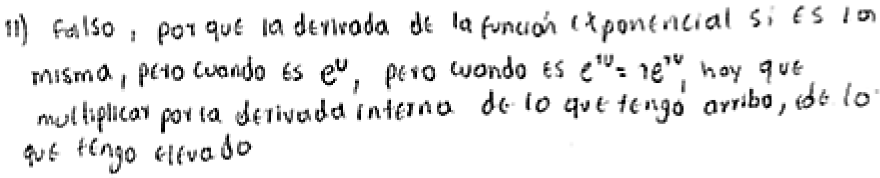
*Solution A. Pre-service*


Esta mulo No aplicaron la Regla de la cadena
de $D_x e^u = e^u * du$

*It is wrong; the students did not apply the chain rule.*

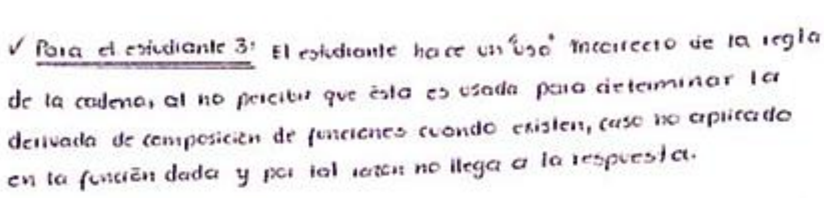
**Figure 16 (cont'd)**

*Solution B. Pre-service*


11) Falso, porque la derivada de la función exponencial si es la misma, pero cuando es $e^u$, pero cuando es $e^{ru} = re^{ru}$, hay que multiplicar por la derivada interna de lo que tengo arriba, de lo que tengo elevado

*False, because the derivative of the exponential function is, indeed, the same, but when it is $e^u$, but when it is $e^{ru}=re^{ru}$, you have to multiply by the inner derivative, of what I have above, of what I have squared…*

*Solution C. In-service*


✓ Para el estudiante 3: El estudiante hace un 'uso' incorrecto de la regla de la cadena, al no percibir que ésta es usada para determinar la derivada de composición de funciones cuando existen, caso no aplicado en la función dada y por tal razón no llega a la respuesta.

*The student 'use', incorrectly, the chain rule, he does not perceive that the chain rule is used to calculate the derivative of a composite function, when* *it exists. The case is not applied to the given function, and based on it the student does not obtain the answer.*

**Table 24**

*Difficulty index of items of questionnaire*

| Task-Item | Difficulty Index pre-service | Difficulty Index in-service | Pre-service % | In-service % |
|---|---|---|---|---|
| 1 | 0.18 | 0 | 18 | 0 |
| 2-a | 0.36 | 0 | 36 | 0 |
| 2-b | 0.5 | 0 | 50 | 0 |
| 2-c | 0.45 | 0 | 45 | 0 |
| 2-d | 0.77 | 0 | 77 | 0 |
| 3-a | 0.36 | 0 | 36 | 0 |
| 3-b | 0.63 | 0 | 63 | 0 |
| 4-a | 0.63 | 0 | 63 | 0 |
| 4-b | 0.86 | 0 | 86 | 0 |
| 5-a | 0.63 | 0.18 | 63 | 18 |
| 5-b | 0.81 | 0.09 | 81 | 9 |
| 6-a | 0.86 | 0.27 | 86 | 27 |
| 7 | 0.95 | 0.27 | 27 | 27 |

According to the study results by Turnuklu and Yesildere (2007), teachers had a deep understanding of mathematical knowledge, but that was not sufficient for them to perform all the stages of error analysis satisfactorily, much less providing remedial strategies. In their research on teacher knowledge for error analysis, Peng, and Luo (2009) used the four error phrases identify, interpret, evaluate, and remediate. In their study, they found out that teachers managed to identify a student’s error, ‘but interpreted it with wrong mathematical knowledge, which led to a meaningless assessment of the student’s performance and unspecific presentation of teaching strategy’ (p. 24). In general, the questionnaire applied to pre-service and in-service teachers had an average difficulty, as illustrated in Table 24.

For the prospective teachers, the most difficult items of the questionnaire were 2-d; 2-c; 3-a; 3-b; 4-a; 4-b; 5-b; 6-a. The solutions for this task require either establishing links among mathematics objects or use non-algebraic strategies. For in-service teachers, we could say that none of the exercises were too difficult for them to solve. In what follows, the complex index (number of incorrect solutions plus number of no responses divided by the number of informants) are displayed for each task.

Due to the objective for Task 8 - assess how teachers evaluate an incorrect solution- we do not consider the difficulty index, nonetheless, we outline that 68% of the pre-service teachers considered that the wrong solution provided by ‘would-be’ students was correct, while 63% of in-service teachers considered that solution as incorrect but identify the ‘would-be’ students’ error.

## FINAL REFLEXIONS

The results obtained from quantitative and qualitative analysis of solutions that respondents gave to tasks in the questionnaire point out that the prospective teachers express specific difficulties when solving tasks related to the common, extended, and epistemic facet of the DMK on derivative. In contrast, the results for the same items show that the questions are not so challenging for in-service teachers.

Task 4 results reveal that the respondents have a better performance when the derivative is used as the slope of the straight-line tangent. Moreover, based on the results presented in Table 11, we can observe that the prospective teachers had problems to demonstrate, by using formal derivative definition, the proposition “the derivative of a constant function is always zero”, which suggests that those students are not well acquainted with proof when it requires the use of the derivative as limit of average rates of variation. In contrast, we can also see that the results for in-service teachers are the opposite; thus, all in-service teachers show competence not only in solving the tasks but explaining them in a way that can be assessed as correct.

The results obtained in Tasks 6 and 7 show the pre-service teachers' difficulties when using the derivative as the instantaneous rate of change. The questionnaire has allowed identifying facets of didactic-mathematical knowledge brought into play to solve textbooks' tasks. It has made evident how common content knowledge is not enough to deal with tasks for teaching, for which not only some level of epistemic facet knowledge but also extended content knowledge is required. We noticed both insufficiencies in the epistemic facet and extended content knowledge, shown by prospective teachers, and an apparent lack of connection among the different derivative meanings (Tasks 1 and Task 5). The answers of the prospective teachers show the complex framework of mathematical practices, objects, and processes brought into play when solving tasks related to the derivative. The awareness of this complexity is necessary to develop and evaluate the mathematical competency of their future pupils.

The results for Task 8 inform that the pre-service teachers did not identify errors nor their nature and ways of explaining them to the 'would-be students." It would be interesting to design tasks that put to test the knowledge required not only to teach mathematics but to identify meaning conflicts and ways to deal with them.

Comparing the results for the pre-service teachers with those of in-service teachers, we can conclude that the in-service teachers manifest an ampler common content knowledge and extended content knowledge and the epistemic facet of the DMK. The knowledge of the pre-service teachers could be dubbed as 'insufficient,' perhaps due to the type of education received in university classrooms: intra-mathematical tasks, traditional teaching, and learning strategies aimed at the formalisation of mathematical notions. While in the case of the in-service teachers, perhaps the experience they acquired during their teaching experience in the field in which they work (classes in careers such as economics, engineering) allows them to have a broader knowledge of relationships between the different meanings of the derivative and the uses in intra and extra-mathematical contexts. Likewise, we cannot conclude whether such experiences allow service teachers to propose richer justifications for their solutions, which would contribute to improving teaching to their future students if achieved by the pre-service teachers.

Although in-service teachers' knowledge should be assessed before developing education programmes, it is also imperative that this knowledge be compared to in-service teachers' knowledge, particularly in the light of research of teachers' knowledge models to identify similarities and differences and to understand how mathematics knowledge is transformed into didactic-mathematics knowledge and to identify connections teachers proposed. It seems that teaching experience is the factor that affect the most the way teachers solve, explain students' errors, and offer ensuing explications. Knowing the exact teacher knowledge and the kind of epistemic configurations used by the teachers would favour assessing the adequacy of such knowledge to teach mathematics. Suzuka et al. (2009) constructed tasks for teacher education and professional development to develop teachers' mathematical knowledge entailed in teaching; according to their results, it is crucial to design such task having in mind both pre-service and in-service teachers due to the similarities in their education and working places culture and curriculum.

The insufficiencies shown in pre-service teachers justify the pertinence to design specific formative actions to develop prospective teachers' epistemic facet of didactic-mathematical knowledge, notably, the meanings of derivatives used according to the contexts. In fact, in such formative cycles, we recommend that both in-service and pre-service teachers discuss activities in which they can identify and reflect on possible mistakes and errors made by students or colleagues. Indeed, the development of these formative cycles should consider the complexity of the global meaning of derivative (Pino-Fan, Godino, & Font, 2018) and propose a way for the pre-service teachers to learn to notice (Castro, Pino-Fan, & Velásquez-Echavarría, 2018). Nonetheless, the epistemic facet of teacher's knowledge points out not only teachers' meanings of derivative but to their teaching practices in their formative courses.

Finally, we think that future lines and research that contribute to the understanding of the components and characteristics of the teachers' didactic-mathematical knowledge on derivatives would be related to the creation of spaces for reflection on practice (planning, implementation, and evaluation), on the discourse and the acquisition of argumentative competencies, and skills for the analysis of the students' mathematical activity.

## ACKNOWLEDGMENTS

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## AUTHORS' CONTRIBUTIONS STATEMENTS

LPF designed seven questions out of eight included in the questionnaire, and WFC supervised the project. LPF developed the theory. WFC and LPF adapted the methodology to this context, WFC applied the questionnaire and collected the data. WFC and LPF analysed the data. All authors discussed the results and contributed to the final version of the manuscript.

## DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by Walter F. Castro G, upon reasonable request.

## REFERENCES

Akkoç, H., Bingolbali, E., & Ozmantar, F. (2008). Investigating the technological pedagogical content knowledge: A case of derivative at a point. In O. Figueras, & A. Sepúlveda (Eds.), *Proceedings of the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the XXX North American Chapter* (Vol. 2, pp.17-24). PME.

Amit, M. & Vinner, S. (1990). *Some misconceptions in calculus. Anecdotes or the tip of an iceberg?* En G. Booker, P. Cobb y T. de Mendicuti (Comps.) *Proceedings of the fourteenth PME Conference*, vol. 1 (pp. 3-10). Comité de Programa.

An, S. & Wu, Z. (2012). Enhancing mathematics teachers' knowledge of students' thinking from assessing and analysis misconceptions in homework. *International Journal of Science and Mathematics Education*, *10*, 717-753.

Asiala, M., Cottrill, J., Dubinsky, E. & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivate. *Journal of Mathematical Behavior*, *16*, 399-431.

Badillo, E. (2003). *La derivada como objeto matemático y como objeto de enseñanza y aprendizaje en profesores de matemáticas de Colombia.* Ph.D. Thesis, Universitat Autònoma de Barcelona, Spain.

Badillo, E., Azcárate, C., & Font, V. (2005). Conflictos semióticos relacionados con el uso de la notación incremental y diferencial en libros de física y de matemáticas del bachillerato. *Enseñanza de las Ciencias. Extra number. VII Congress*, 1-6.

Badillo, E., Azcárate, C., & Font, V. (2011). Análisis de los niveles de comprensión de los objetos $f'(a)$ y $f'(x)$ en profesores de matemáticas. *Enseñanza de las Ciencias, 29*(2), 191-206.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*, 241-247.

Ball, D. L., & Bass, H. (2009). With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures. *Paper prepared based on keynote address at the 43rd Jahrestagung für Didaktik der Mathematik held in Oldenburg*, March 1-4, Germany.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). *Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge*. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). American Educational Research Association.

Baumert, Jürgen & Kunter, Mareike & Blum, Werner & Brunner, Martin & Voss (Dubberke), Thamar & Jordan, Alexander & Klusmann, Uta & Krauss, Stefan & Neubrand, Michael & Tsai, Yi-Miau. (2010). *Teachers' Mathematical Knowledge, Cognitive Activation in the Classroom, and Student Progress*. American Educational Research Journal, 47, 133-180. 47. http://doi.org/10.3102/0002831209345157

Berkeley, S., Scruggs, T. E., & Mastropieri, M. A. (2010). Reading comprehension instruction for students with learning disabilities, 1995-2006: A meta-analysis. *Remedial and Special Education, 31*(6), 423-436. http:/doi.org/10.1177/0741932509355988

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology, 29*, 389-399.

Bingolbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics, 68*(1), 19-35. http://doi.org/10.1007/s10649-007-9112-2

Brigham, F. J., Scruggs, T. E., & Mastropieri, M. A. (2011). Science education and students with learning disabilities. *Learning Disabilities Research & Practice*, *26*(4), 223-232.

Castro, W. F., Pino-Fan, L., & Velásquez-Echavarría, H. (2018). A proposal to enhance preservice teacher's noticing. *Eurasia Journal of Mathematics, Science & Technology Education*, *14*(11), em1569. http://doi.org/10.29333/ejmste/92017

Carney, R. N., & Levin, J. R. (2000). Mnemonic instruction, with a focus on transfer. *Journal of Educational Psychology*, *92*(4), 783-790. http://doi.org/10.1037/0022-0663.92.4.783

Çetin, N. (2009). The ability of students to comprehend the function-derivative relationship with regard to problems from their real life. *PRIMUS*, *19*(3), 232-244.

De Almeida, L. M. W., & Da Silva, K. A. P. (2018). A semiotic interpretation of the derivative concept in a textbook. *ZDM*, *50*, 881-892.

Desfriti, R. (2016). In-service teachers' understanding on the concept of limits and derivatives and the way they deliver the concepts to their high school students. *Journal of Physics: Conference Series,* 693. http://doi.org/10.1088/1742-6596/693/1/012016

Delos Santos, A. (2006). *An investigation of students' understanding and representation of derivative in a graphic calculator-mediated teaching and learning environment*. Unpublished Ph. D. Thesis, University of Auckland, New Zealand.

Dreyfus, T., & Eisenberg, T. (1990). On difficulties with diagrams: Theoretical issues. *Proceedings of the 14th International Conference for the Psychology of Mathematics Education*, México, (p. 27-34).

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics*, *82*, 97-124.

Furinghetti, F., & Paola, D. (1991). The construction of a didactic itinerary of calculus starting from students' concept images (ages 16-19). *International Journal of Mathematical Education in Science and Technology*, *22*, 719-729. http://doi.org/10.1080/0020739910220503

García, L., & Azcárate, C. & Moreno, M. (2006). Creencias, concepciones y conocimiento profesional de profesores que enseñan cálculo diferencial a estudiantes de ciencias económicas. *Revista Latinoamericana de Investigación en matemática Educativa (RELIME)*, *9*(1), 85-116.

García, M., Llinares, S., & Sánchez-Matamoros, G. (2011). Characterizing thematized derivative schema by the underlying emergent structures. *International Journal of Science and Mathematics Education*, *9*, 1023-1045.

Gökçek, T., & Açıkyıldız, G. (2015). Preservice mathematics teachers' errors related to derivative. *Turkish Journal of Computer and Mathematics Education*, *7*(1), 112-141. http://doi.org/10.16949/turcomat.14647

Godino, J. D. (2009). Categorías de análisis de los conocimientos del profesor de matemáticas. *Unión, Revista Iberoamericana de Educación Matemática*, *20*, 13-31.

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, *39*(1), 127-135.

Godino, J. D., Font, V., Wilhelmi, M., & Lurduy, O. (2011). Why is the learning of elementary arithmetic concepts difficult? Semiotic tools for understanding the nature of mathematical objects. *Educational Studies in Mathematics*, *77*(2), 247-265.

Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. Teachers College Press.

Habre, S., & Abbud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior*, *25*, 52-72.

Haciomeroglu, E., Aspinwall, L., & Presmeg, N. (2010). Contrasting cases of calculus students' understanding of derivative graphs. *Mathematical Thinking and Learning*, *12*(2), 152-176.

Hänkiöniemi, M. (2006). Is there a limit in the derivative? Exploring students' understanding of the limit of the difference quotient. In: M. Bosch (Ed.), *Proceedings of the fourth congress of the European society for research in mathematics education* (CERME 6) (pp. 1758-1767), Sant Feliu de Guixols, Spain. European Research in Mathematics Education.

Hatisaru, V., & Erbas, A.K. (2017). Mathematical Knowledge for Teaching the Function Concept and Student Learning Outcomes. *Int Journal of Sci and Math Educ*, *15*, 703-722. http://doi.org/10.1007/s10763-015-9707-5

Hauger, G. S. (2000). Instantaneous rate of change: A numerical approach. *International Journal of Mathematical Education of Science and Technology*, *31*(6), 891-897.

Heid, K. M. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education*, *19*(1), 3-25.

Hill, Heather & Blunk, Merrie & Charalambous, Charalambos & Lewis, Jennifer & Phelps, Geoffrey & Sleep, Laurie & Ball, Deborah. (2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction*, *26*. 430-511. http://doi.org/10.1080/07370000802177235.

Hill, H. C., Ball, D. L., & Schlling, S. G. (2008). Unpacking pedagogical content knowledge of students. *Journal for Research in Mathematics Education, 39,* 372-400.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*(4), 430-511.

Inglada, N., & Font, V. (2005). Significados institucionales y personales de la derivada. Conflictos semióticos relacionados con la notación incremental. *Proceedings of the V Congreso Internacional Virtual de Educación (CIVE'05)*, Córdoba. (p. 1-16).

Johnson, R. B., & Onwuegbuzie, A. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Research*, *33*(7), 14-26.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, *6*(3), 223–252.

Kunter, M., Klusmann, U., Baumert, J., Richter, D., Voss, T., & Hachfeld, A. (2013). Professional competence of teachers: Effects on instructional quality and student development. *Journal of Educational Psychology*, *105*(3), 805–820.

Lam, T. (2009). On In-Service Mathematics Teachers' Content Knowledge of Calculus and Related Concepts. *The Mathematics Educator*, *12*(1), 69-86.

Letendre, W. (1993). Mnemonic instruction with regular and special education students in social studies. *Southern Social Studies Journal*, *18*(2), 25-37.

Malaspina, U., & Font, V. (2010). The role of intuition in the solving of optimization problems. *Educational Studies in Mathematics*, *75*(1), 107-130.

Maschietto, M. (2008). Graphic calculators and micro-straightness: analysis of a didactic engineering. *International Journal of Computers for Mathematical Learning*, Netherlands, *13*(3), 207-230.

Mhlolo, M. (2012). Mathematical connections of a higher cognitive level: a tool we may use to identify these in practice. *African Journal of Research in Mathematics, Science and Technology*, *16*(2), 176-191.

Moon, K., Brenner, M., Jacob, B., & Okamoto, Y. (2013). Prospective secondary mathematics teachers' understanding and cognitive difficulties in making connections among representations. *Mathematical Thinking and Learning*, *15*(3), 201-227.

Muhundan, A. (2005). *Effects of using graphing calculators with a numerical approach on students' learning of limits and derivatives in an applied calculus course at a community college*. http://scholarcommons.usf.edu/etd/780/

Orhun, N. (2012). Graphical understanding in mathematics education: Derivative functions and students' difficulties. *Procedia - Social and Behavioral Sciences*, *55*(2012), 679-684.

Orton, A. (1983). Students' understanding of differentiation. *Educational Studies in Mathematics*, *14*, 235-250.

Otero, M. R. & Llanos, V. C. (2019). Difficulties faced by the in-service mathematics teachers planning lessons based on questioning during a training course. *International Journal of Research in Education and Science (IJRES)*, *5*(2), 429-436.

Ozmantar, M. F., Akkoç, H., Bingolbali, E., Demir, S., & Ergene, B. (2010). Pre-service mathematics teachers' use of multiple representations in technology-rich environments. *Eurasia Journal of Mathematics, Science & Technology Education, 6*(1), 19-36.

Panero, M., Arzarello, F., & Sabena, C (2016). The mathematical work with the derivative of a function: Teachers' practices with the idea of "generic". *Bolema: Boletim de Educação Matemática, 30*(54), 265-286. http://doi.org/10.1590/1980-4415v30n54a13

Park, J. (2015). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology*, *44*(5), 624-640.

Peng, A., & Luo, Z. (2009). A framework for examining mathematics teacher knowledge as used in error analysis. *For the Learning of Mathematics, 29*(3), 22–25.

Pino-Fan, L., Assis, A., & Castro, W. F. (2015). Towards a methodology for the characterization of teachers' didactic-mathematical knowledge. *Eurasia Journal of Mathematics, Science & Technology Education, 11*(6), 1429-1456. http://doi.org/10.12973/eurasia.2015.1403a

Pino-Fan, L., Godino, J. D., Font, V., & Castro, W. F (2013). Prospective teacher's specialized content knowledge on derivative. In B. Ubuz, Ç. Haser & M. Mariotti (Eds.), *Proceedings of the Eighth Congress of European Research in Mathematics Education* Antalya, Turkey. (pp. 3195-3205). CERME.

Pino-Fan, L., Godino, J. D., & Font, V. (2018). Assessing key epistemic features of didactic-mathematical knowledge of prospective teachers: the case of the derivative. *Journal of Mathematics Teacher Education,* 21(1), 63-94. http://dx.doi.org/10.1007/s10857-016-9349-8

Rogalski, M. (2008). Les rapports entre local et global: mathématiques, rôle en physique élémentaire, questions didactiques. In: Viennot, L. (Ed.). *Didactique, épistémologie et histoire des sciences*. (p. 61-87). Presses Universitaires de France.

Rosnawati, R., Wijaya. A & Tuharto (2020). The perspective prospective professional teachers toward (specific) pedagogical content knowledge on derivative concept. *Journal of Physics: Conference Series, 1581*. doi.org/doi:10.1088/1742-6596/1581/1/012050

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*(3), 255-281.

Sahin, Z., Yenmez, A. A., & Erbas, A. K. (2015). Relational understanding of the derivative concept through mathematical modeling: A case study Eurasia. *Journal of Mathematics, Science & Technology Education, 2015*, *11*(1), 177-188. http://doi.org/10.12973/eurasia.2015.1149a

Sánchez-Matamoros, G., Fernández, C., & Llinares, S. (2014). Developing pre-service teachers' noticing of students' understanding of the derivative concept. *International Journal of Science and Mathematics Education*, *13,* 1305-1329. http://doi.org/10.1007/s10763-014-9544-y

Saltan, F., Arslan K., & Wang, S. (2017). A comparison of in-service and preservice teachers' technological pedagogical content knowledge self-confidence. *Cogent Education*, *4*(1), article 1311501.

Santi, G. (2011). Objectification and semiotic function. *Educational Studies in Mathematics*, *77*(2-3), 285-311.

Schoenfeld, A., & Kilpatrick, J. (2008). Towards a theory of proficiency in teaching mathematics. En D. Tirosh, & T. L. Wood (Eds.), *Tools and processes in mathematics teacher education*. (pp. 321-354). Sense Publishers.

Scruggs, T. E., Mastropieri, M. A., Berkeley, S., & Graetz, J. E. (2009). Do special education interventions improve learning of secondary content? A meta-analysis. *Remedial and Special Education*, *31*(6), 437-449.

Selden, J., Mason, A., & Selden, A. (1989). Can average calculus students solve non-routine problems? *Journal of Mathematical Behavior*, *8*, 45-50.

Sharplin E., Peden S., & Marais, I. (2016). Making the grade: Describing inherent requirements for the initial teacher education practicum. *Asia-Pacific Journal of Teacher Education*, *44*(3), 224-241. http://doi.org/10.1080/1359866X.2015.1105933

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), 1-22.

Silverman, J., & Thompson, P. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*(6), 499-511. http://doi.org/10.1007/s10857-008-9089-5

Suzuka, K., Sleep, L., Ball, D. L., Bass, H., Lewis, J. M., & Thames, M. H. (2009). Designing and using tasks to teach mathematical knowledge for teaching. In D. Mewborn & H. S. Lee (Eds.), *Scholarly practices and inquiry in the preparation of teachers. Association of Mathematics Teacher Education Monograph Series* (Vol. 6, pp. 7-23). Association of Mathematics Teacher Education.

Tall, D. (1991). Intuition and Rigor: The role of visualization in the calculus. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics*. (pp. 105–119). Mathematical Association of America.

Tsamir, P., Rasslan, S., & Dreyfus, T. (2006). Prospective teachers' reactions to Right-or-Wrong tasks: The case of derivatives of absolute value functions. *Journal of Mathematical Behavior*, *25*, 240-251.

Turnuklu, E. B., & Yesildere, S. (2007). The pedagogical content knowledge in mathematics: Preservice primary mathematics teachers' perspectives in Turkey. *IUMPST: The Journal*, *1*, 1-13.

Ubuz, B. (2001). First year engineering students' learning of point of tangency, numerical calculation of gradients, and the approximate value of a function at a point through computers. *Journal of Computers in Mathematics and Science Teaching*, *20*(1), 113–137.

Vandebrouck, F. (2011). Students' conceptions of functions at the transition between secondary school and university. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Conference of European Researchers in Mathematics Education*, Rzeszow. (pp. 2093-2102).

Viholainen, A. (2008). Finnish mathematics teacher student's informal and formal arguing skills in the case of derivative. *Nordic Studies in Mathematics Education*, *13*(2), 71-92.

Zazkis, R. (2008). Examples as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *International handbook in mathematics teacher education, vol. 2: Tools in mathematics teacher education* (pp. 135–156). Sense.

# APPENDIX

**Figure 1**

*Task 1*

Task 1

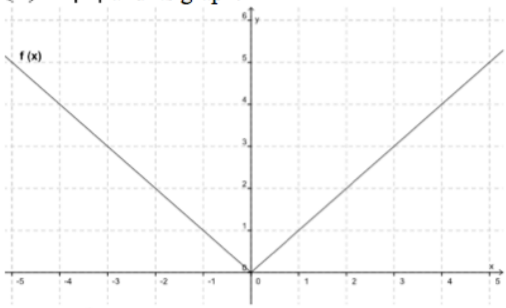
What does the derivative mean to you?

**Figure 2**

*Task 2*

Task 2

Analyze the function $f(x) = |x|$ and its graph.




a) For which values of *x*, $f(x)$ is derivable?

b) If it is possible, calculate $f'(2)$ and draw a graphic representation of your solution. If it is not possible explain why.

c) If it is possible, calculate $f'(0)$ and draw a graphic representation of your solution. If it is not possible explain why.

d) Based on the definition of the derivative, justify why does the graphic of a derivable function cannot have "peaks" (corners, angles).

**Figure 3**

***Task 3***

Task 3

For a given function $y = f(x)$, the values of the table as follows are fulfilled:

| $x$ | $f'(x)$ |
|---|---|
| 0 | 0 |
| 1.0 | 2 |
| 1.5 | 3 |
| 2.0 | 4 |
| 2.5 | 5 |

a) Find an expression for $f(x)$

b) Can you find a second expression, different from the previous one for $f(x)$? Which would this be? Justify your answer.

**Figure 4**

*Task 4*

Task 4

a) By using graphic representations, explain: Why does the derivative of a constant function is always zero?

b) By using the formal definition of the derivative, demonstrate that the derivative of a constant function is zero.

*Figure 5*

**Task 5**

Task 5

Given the function $y = x^3 - \frac{x^2}{2} - 2x + 3$

a) Find the points of the graph of the function for which their tangent is horizontal.

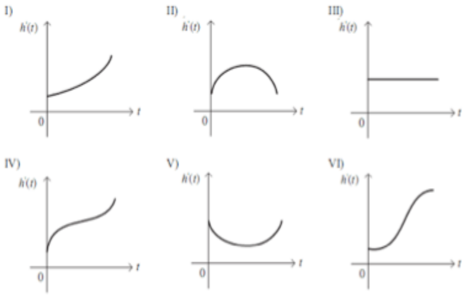
b) In which points the instantaneous rate of change of y with respect to x is zero?

*Figure 6*

**Task 6**

Task 6.

The filling flow of the cups R, S and T of the figure, is constant. The water high *h* inside the cups is a function of time. The graphs of six functions h'(t) are given below; three of them correspond to derivatives of functions of cups' filling.


a) Relate each cup to the corresponding derivative graph. Explain your reasoning for each relation.
b) Which abilities are brought into play for solving this problem?

*Figure 7*

**Task 7**

Task 7

A ball is shot to the air from a bridge with 11 meters high. $f(t)$ denotes the ball's distance from the ground in a $t$ time Some values $f(t)$ are collected in the table as follows:

| $t$ (sec.) | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|---|---|---|---|---|---|---|---|
| $f(t)$(m.) | 11 | 12.4 | 13.8 | 15.1 | 16.3 | 17.4 | 18.4 |

According to the table, What is the speed of the ball when it reaches a high in $t = 0.4$ seconds? Justify your answer's choice.

a) 11.5 m/s  b) 1.23 m/s  c) 14.91 m/s  d) 16.3 m/s  e) Other

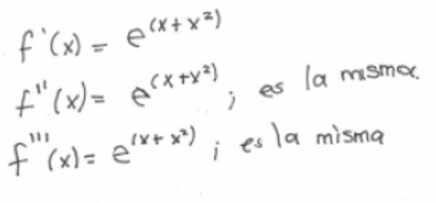
**Figure 8**

*Task 8*

Un profesor dio este problema a sus estudiantes:

Calcular las tres primeras derivadas de la siguiente función: $f(x) = e^{(x+x^2)}$

Las respuestas que varios estudiantes dieron fueron estas:




Interrogados por sus respuestas afirmaron: *Bien, la derivada de la función exponencial es ella misma…*

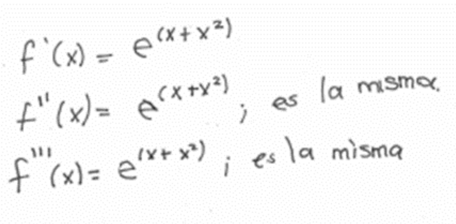
¿Qué explicación daría Usted a los estudiantes?

Transcription:

A teacher proposed the following problem to his students:

Calculate the first tree derivatives of the following function:

Students' answer are:




Asked by their solutions the students said 'well, the derivative of an exponential function is the same'.

What explanation would you offer to the students?