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A Didactic Engineering in the Research Process of the Generalization of the Padovan sequence: An Experience in a Pre-Service Teacher Training Course

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ABSTRACT

Background: Obstacles are found during the epistemological construction of mathematical concepts research, aiming to contribute to the Didactics of Mathematics through a study of Padovan sequence. **Objectives**: describe elements of a systematic study, based on Didactic Engineering in conjunction with the Theory of Didactic Situations. I addition, referring to the generalization model of Padovan sequence and promoting a historical-evolutionary understanding and its mathematical properties. **Design:** it presents the most representative data of an investigation supported by the foundations of Didactic Engineering research design, in association with the Theory of Didactic Situations teaching methodology. **Setting and Participants:** the research was developed in 2019 and applied in a Pre-Service Mathematics Teacher Training Course in the History of Mathematics discipline, with the eight students enrolled. **Data collection and analysis:** data validation occurred internally due to the short period of the research. **Results:** it describes an investigation around the object of study, the Padovan sequence, focusing on the generalization process of this sequence and its properties. Thus, three problem situations are elaborated and analyzed based on the assumed research and teaching methodologies, seeking to examine their properties and the student's intuitive thinking, before the insertion of a historical-epistemological conception of this investigation. **Conclusions:** the research makes it possible to extract repercussions, suggest and promote research scripts aiming at the formation of teachers (initial) in the context of the teaching of History of Mathematics.

Keywords: Didactic Engineering; History of mathematics; generalization; Padovan sequence; Theory of Didactic Situations.

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Uma Engenharia Didática no Processo de Investigação da Generalização da Sequência de Padovan: Uma Experiência em um Curso de Licenciatura

RESUMO

Contexto: Obstáculos são encontrados durante pesquisas referentes a construção epistemológica de conceitos matemáticos, contribuindo com a Didática da Matemática por meio de um estudo da sequência de Padovan. **Objetivos:** descrever elementos de um estudo sistemático, fundamentado na Engenharia Didática em conjunto com a Teoria das Situações Didáticas, referente ao modelo de generalização da sequência de Padovan, promovendo ainda uma compreensão histórico-evolutiva e, as suas propriedades matemáticas. **Design:** apresenta os dados mais representativos de uma investigação amparada pelos fundamentos de uma Engenharia Didática, utilizada como design de pesquisa, em associação com a Teoria das Situações Didáticas, como metodologia de ensino. **Ambiente e participantes:** a pesquisa foi desenvolvida em 2019 e aplicada no curso de Licenciatura em Matemática na disciplina de História da Matemática, com os oito estudantes matriculados. **Coleta e análise de dados:** a validação dos dados ocorreu de forma interna, devido ao curto período temporal da pesquisa. **Resultados:** descreve uma investigação em torno do objeto de estudo, denominado de sequência de Padovan, com enfoque no processo de generalização dessa sequência e de suas propriedades. Assim, são elaboradas três situações-problema e analisadas com base nas metodologias de pesquisa e ensino assumidas, buscando examinar as suas propriedades e o pensamento intuitivo do estudante, diante da inserção de uma concepção histórica-epistemológica dessa investigação. **Conclusões:** a pesquisa possibilita extrair repercussões, sugerir e promover roteiros de investigação visando a formação de professores (inicial) no âmbito do ensino de História da Matemática.

Palavras-chave: Engenharia Didática; História da Matemática; generalização; sequência de Padovan; Teoria das Situações Didáticas.

INTRODUCTION

Currently, there are many studies about the teaching/learning of mathematical concepts and their corresponding position in the pedagogical planning, thus contributing to the Didactics of Mathematics (Oliveira & Alves, 2019). Many French researchers consider this teaching a scientific field, with aligned methodologies, studying the dimension of mathematical knowledge through didactic systems associated with teaching and learning processes (Artigue, 2015). Hence, there are also countless investigations about the existing obstacles in its epistemological construction.

Considering the panorama above, this article presents a survey carried out on History of Mathematics books that include the Fibonacci sequence as content, referring to the linear and recurring sequences. However, this sequence has its relation of convergence as being the gold number, where the gold number and the plastic number are the only two solutions of the morphic numbers (Ferreira, 2015; Spinadel & Buitrago, 2009). Thus, the researchers sought a sequence which presented its relation of convergence related to the plastic number, discovering, then, the Padovan sequence.

In this context, this study addressed the historical and investigative evolution of the Padovan sequence, transforming it into content to be taught. During the process of teaching mathematical concepts related to our object of study, there may be some obstacles that can be overcome through the sequence evolution process, with an emphasis on its generalization.

For this, we consider the epistemological, cognitive and didactic aspects, aiming to structure this investigative study based on the methodology or research design of Artigue's (1988) Didactic Engineering (DE) as a complement to Brousseau's (1976) Theory of Didactic Situations (TDS) around the historical context of Padovan numbers.

We begin this discussion by introducing the rationale for this work, followed by the previous analyses supported by the DE. We justify this research based on the following guiding question: how can we develop didactic situations that allow us to investigate the Padovan numbers, theorems and properties, while also exploring the matrix representation in an epistemological perspective, in relation to its origin and historical evolution?

Thus, offering the necessary tools to improve classroom practices, we intend to present a didactic-mathematical approach to mathematics teachers in initial training. To overcome any obstacles, we outlined general and specific objectives, highlighting elements of an epistemological, cognitive, and didactic nature, based on the DE. Therefore, the general objective of the research is to describe elements of a systematic study, based on the DE together with the TDS, related to the generalization model of Padovan numerical sequence, fostering a historical-evolutionary understanding, and exploring its mathematical properties (Alves, 2015; 2017).

From this general objective, we drew the specific objectives, namely: 1) to investigate the theorems and properties inherent to the generalization of matrix representations, Binet formula for the Padovan sequence; 2) to explore, through teaching situations, the process of generalization of the Padovan sequence; 3) to apply the historical-evolutionary development of the Padovan sequence in relation to its generalization process, through problem situations in the classroom.

This work is an excerpt from the research carried out in the Master's Program of the Graduate Program in Science and Mathematics Teaching (PGECM) at the Federal Institute of Education, Science and Technology of the State of Ceará (IFCE), based on the DE research methodology together with the TDS teaching methodology. This article was approved by the Research Ethics Committee (CEP, in the Portuguese acronym) (review: 3.314.835). The research elaborated structured teaching situations, motivating students to develop intuitive reasoning during the understanding of the mathematical concepts addressed.

PRELIMINARY ANALYSIS

Based on the DE, the preliminary analyses of the research were divided into two stages. In the first stage, we conducted a systematic bibliographic survey of the research methodology used (DE), the teaching methodology, TDS, and the object of study, i.e., the

Padovan sequence. In the second stage, we used a didactic-methodological approach to the Padovan sequence, correlating it with its application and planning and development in the classroom. For example, Maschietto (2008) analyzed the different ways we can use a given function: local, specific, global, local notion of rectitude and way of thinking. The epistemological analyses developed allowed us to understand the language used and mobilized by the subjects participating in the study, identifying the use of mathematical rules.

When building the theoretical framework, we delved mathematically into the Padovan numbers to later develop the historical-evolutionary process, emphasizing the generalization of its initial terms and the coefficients of the recurrence formula, based on the work of: Vieira and Alves (2018; 2019), Seenukul, Netmanee, Panyakhun, Auiseekaen, and Muangchan (2015), Sokhuma (2013), Gulec and Taskara (2012), Spinadel and Buitrago (2009), Civciv and Turkmen (2008) and Stewart (1996). The development of the generalization of the Padovan sequence was based on the scientific works described and some of the mathematical content presented shows some degree of novelty, since some properties discussed in this research have not been found in the scientific literature until the present moment.

Finally, we conducted analyses based on the DE and TDS methodologies, aiming to investigate teaching situations, from the perspective of the French Didactics of Mathematics, highlighting other research developed in a similar context to ours: Oliveira, Alves e Silva (2019), Vieira, Alves and Catarino (2019), Alves and Dias (2018), Artigue (2018; 2015; 2014; 1995; 1988), Oliveira (2018), Alves and Catarino (2017), Santos (2017), Almouloud (2016; 2007), Perrin-Glorian and Bellemain (2016), Alves (2016; 2014), Kidron (2014), Teixeira and Passos (2013), Margolinas (2012), Oliveira and Araújo (2012), Brousseau (2008; 2002; 2000; 1997; 1986; 1982; 1976), and Laborde (1997).

Also in the second stage of the preliminary analyses, we examined the curriculum matrix and course syllabuses of the Pre-Service Mathematics Teacher Training Course at the Federal Institute of Education, Science and Technology of Ceará (IFCE), in 2019, selected because it offers an initial training course for math teachers. One mandatory subject matter, History of Mathematics, was chosen as the most suitable for this research as it covers issues inherent to the concept of numbers and the numbering system, presents the biography of mathematicians who contributed to the historical process of mathematics in Brazil, and carries out a historical study of numerical and recurring sequences (Alves, 2015; Alves; Vieira; Catarino & Mangueira, 2020).

Although the sequence content is studied in the section that discusses the History of Mathematics, it presents only the Fibonacci sequence, leaving aside the Padovan sequence. Also, none of the books adopted in that subject matter at the institution addresses the Padovan numbers, which can be found only on the *Internet*, in scientific articles in pure mathematics, published in national and international newspapers and journals, among others.

DIDACTIC ENGINEERING

Aiming to improve some practices in the education system, the DE appeared in France in the 1980s, developed to understand phenomena derived from Mathematics teaching and learning. According to Artigue (1988), this methodology represents work comparable to that of an engineer (*l´*ingénieur), who needs to rely on scientific knowledge of their technical domain, being also obliged to work with more complex objects than those filtered by science. With this, the teacher assumes the role of an engineer that plans so that students can understand specific scientific knowledge.

During its emergence, some risks related to transformations in teaching were observed, and researchers wondered whether the system was prepared for such a substantial change, making the role of the teacher an object of study (Perrin-Glorian & Bellemain, 2016). However, DE is currently considered an important research methodology or investigative *design*, developing an expedient of analysis and study of students' and teachers' behavior, with an interest in modeling their activities during the learning process.

Alves and Catarino (2017) also state that DE enables the study of classroom phenomena, developing resources for teacher training. Thus, it is possible to analyze the importance of the teacher's activity and its didactic transposition action (Chevallard, 1991) of scientific knowledge. Those studies developed in the field of the French aspect of Didactics of Mathematics around the interest in the teachers' role consider classroom conditions and scientific experiments (Artigue, 2018). Based on these perspectives, we assume the character of relevance to address the process of generalization of the Padovan sequence by associating the DE and the TDS methodologies in the initial training course for mathematics teachers, emphasizing the math teacher's formative activity and learning.

Therefore, this research adopted the DE methodology, which can be presented through two levels of research, microengineering and macroengineering. The first has a more limited view in relation to the classroom phenomena. The second, on the other hand, portrays a more global view of those phenomena. Taking this categorization into account, this work is based on microengineering, developing a DE that aims at the teaching and generalization of the Padovan sequence. This level of research is considered complex, since the facts and phenomena seen and recorded in the classroom "are not easy to develop in practice" (Artigue, 1995, p. 36).

Studying the relationship between theory and practice, this methodology is then divided into four classic phases: preliminary analysis, design and *a priori* analysis, experimentation and *a posteriori* analysis, and internal validation and external validation (Laborde, 1997). Alves (2014) also considers that the application of the DE phases is indispensable for the process of data investigation and interpretation. In the preliminary analysis, together with identifying the relevant teaching and learning problems, we carried out a bibliographic survey about the theme to be investigated. According to Artigue (1995), this stage is analyzed in three dimensions: in relation to epistemology, comparing it with a game; cognition, being then related to the characteristics of the students under analysis; and didactics, referring to the education system. This phase (of preliminary analysis) was introduced in the previous section, comprising investigations in the inherent literature and the Padovan sequence.

In the second stage, the *a priori* conception and analysis involves the moment when the variables are chosen, which can be of a macrodidactic or microdidactic nature, which will be discussed later, during the design of the didactic situations produced in our study. After choosing the didactic variables, the teacher-researcher must then construct the problem situations according to the epistemic-mathematical field developed, so that they can be applied later with the students, aiming to achieve the research objective. Almouloud (2007) states that:

> *A priori* analysis is extremely important, as the success of the problem situation depends on its quality; besides, it allows the teacher to be able to control the performance of the students' activities, and also to identify and understand the facts observed. Thus, the conjectures that will appear may be considered, and some may be the subject of a scientific debate. (Almouloud, 2007, p.176)

The third phase, experimentation, is when the problem situations developed in the previous phase are applied. Artigue (2015) also comments that during this phase the researchers will be observers, and may also make adaptations to the application *design*, especially when the DE is not evolving as planned. All must be documented and justified to be considered when the *a posteriori* analysis is done.

At that moment, the didactic contract must be established. By this contract, the teacher and the student assume their respective responsibilities, which are, therefore, the behavior the teachers expect from the students and vice versa. There must also be an agreement between the teacher and the students, during the teaching didactic situations in class (Brousseau, 1986). This didactic contract is not always confirmed, since there are cases in which students do not show interest in the activities proposed, resulting in a breach of the contract. On the other hand, since we considered a group of teachers in initial training, we anticipate that there were no disruptions, given the subjects' interest in teaching through the historical context.

Finally, the last phase of the DE, in the *a posteriori* analysis and validation, the data collected during the experimentation are analyzed, being registered through photos, audio recordings, writings, among others. After that, we compare those data with the definitions established in the *a priori* analysis, validating the hypotheses assumed during the investigation, through internal and external validations. The first is an analysis of the students who participated in the research only, while the second analyzes not only the participants of the research, but also those who did not participate in the didactic sequence, depending also on the number of participants (Laborde, 1997). Artigue (2014) also states that the validation of the hypotheses formulated can lead to a collection of data complementary to the data already collected during the experimentation phase, since it is necessary to appreciate the results during the learning process. Thus, data records must be thorough, detailed, and all the processes performed by students in action in the classroom must be recorded.

THEORY OF DIDACTIC SITUATIONS

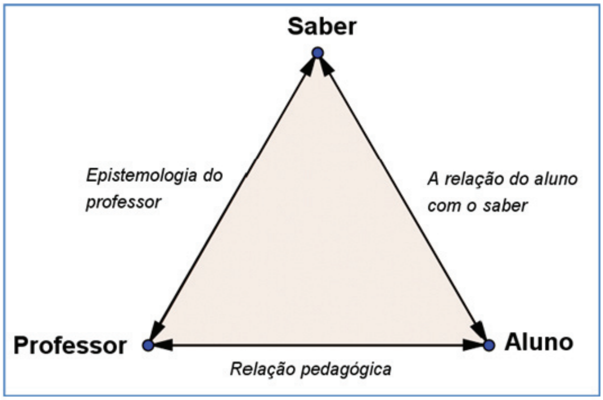
During the experimentation phase of the DE, when teaching sequences are applied, it is necessary to base them on structured teaching situations, so they are directed to the teaching and learning process. In view of this, the TDS, developed by Brousseau, stands out as being a classic theory of the French strand of Didactics of Mathematics, whose main objective involves the modeling of knowledge (Santos, 2017).

In France, this methodology began in the 1960s, and was developed in view of an analysis of the transition between high school and the university *locus*. Thus, French researchers conducted a study about the teaching approach in the Mathematics field, considering socio-cultural and institutional practices (Kidron, 2014). Thus, Brousseau (2000) states that:

> It is about building a model of situations used to introduce or teach mathematical notions (and criticize them), besides imagining more appropriate ones. By placing the problems in this way, it is possible to analyze them, especially their calculation, together with the arguments of the logical-mathematical organization of knowledge, economic and ergonomic arguments. But it is also possible to consider other restrictions, in particular, those that may appear as conclusions of works of psychology or sociology, with the condition of making them functional, that is, of specifying how they intervene effectively. (Brousseau, 2000, p. 11, our translation)

This methodology has as characteristic the elaboration of a set of structured situations that will be applied in the classroom, aiming to investigate the students' behavior during the resolution of the problems proposed. Almouloud (2007) also highlights that, in this teaching methodology, teachers, students and the environment (*milieu*) are essential elements for the teaching and learning relationship to exist. Represented through the didactic triangle, this triad (see Figure 1), has the milieu as the main element, allowing the evolution of knowledge.

Figure 1
Didactic triangle. (Oliveira & Araújo, 2012, p. 214)



It is based on this triad or "classic trinomial", that the exchange of information and the scientific debate are carried out, through a mathematical and coded language, analyzing the students' behavior. Oliveira and Araújo (2012) also consider that the individual must learn according to the adaptation of the environment, so that there are difficulties to be overcome during situations.

A situation, according to Brousseau (2008) is "a model of interaction between a subject and a given environment, gathering the circumstances in which a person finds himself/herself and the relationships that unite him/her to the *milieu*." These situations are elaborated as a kind of game, where the student is motivated to participate in the learning process, developing his own mastery and control of the game, known as didactic situations, which will be used in this research. However, there may still be another situation, in which the student has a basic strategy to assume the knowledge, providing the necessary conditions to acquire the new knowledge, being called an adidactic situation. Having established the notion of situation, the teacher - researcher must analyze the existing elements in the didactic stages of teaching, based on their phases: action, formulation, validation, and institutionalization (Brousseau, 1986). According to Brousseau (2002):

> The sequence of 'action situations' is the process by which the students develop strategies, that is, 'teach themselves' a method of solving their problem. This succession of interactions between the students and the environment, constitutes the dialectic of action. We use the word 'dialectic' instead of the word 'interaction' because, on the one hand, the students are able to anticipate the results of their choices and, on the other hand, their strategies are, in a way, propositions validated or invalidated by experimentation in a kind of dialogue with the situation. (Brousseau, 2002, p. 9, our translation)

At that moment, the students are given an activity and seek, in their previous knowledge, tacit resolutions for a problem situation. The result does not need to follow rules, as they can be readjusted and corrected later. Besides, in this dialectic phase the teacher does not intervene. Alves (2016) considers this phase as the moment when the student strives to solve the activity proposed, producing specific knowledge of a preliminary operational nature. Then, aiming at the proper use of a symbol system in the formulation situation, Vieira, Alves and Catarino (2019) say:

> It can be summarized that during this situation the students will transform the ideas obtained during the action situation, and transform them into a more appropriate language, aiming at carrying out conjectures, theorems, and properties of the mathematical subject. (Vieira, Alves & Catarino, 2019, p. 268)

During this stage, the students present a more formal and more elaborate language, formalizing their attempts at resolutions developed in the previous phase, so that, later, this argument can be validated. Assuming that reasoning is an experiment, we must evolve in each of the phases of this theory, thus applying the information obtained previously.

For the validation situation, Oliveira, Alves, and Silva (2019) recognize that:

> There is validation when the conjectures are checked to be refuted or validated. For this, the student starts to present a more theoretical and formal language, making use of properties and methods of mathematical demonstrations. (Oliveira, Alves & Silva, 2019, p.4)

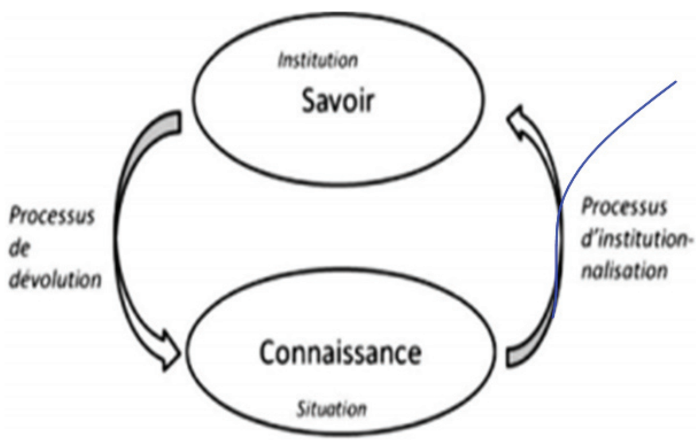
In the validation, the students will use mathematical methods to demonstrate the validity of their resolutions, using a more formal and scientific language, convincing the interlocutors of their arguments. The objective of this stage is to validate/confirm the formulations made during the previous phases of action and formulation, aiming to conclude the scientific debate among students (Alves & Dias, 2018). For the institutionalization situation, Brousseau (1997) states:

> The objective of the different types of situations, whose return we address, is that the students make sense of the pieces of knowledge they manipulate, combining these different components, giving meaning to the knowledge acquired. (Brousseau, 1997, p. 235, our translation)

Teixeira and Passos (2013) report that, at this stage, the teachers take over again the responsibility for the practice, which, until then, had been in the students' hands, verifying the productions developed by the students, formalizing the object of study, revealing their intention and explaining their function. In this last stage, the teacher takes over the situation again, identifying and recognizing the knowledge built in the other stages discussed. The resolutions are checked, and then the real intention of the activity proposed is revealed, evaluating the passage of knowledge (*connaissances*) and scientific knowledge (*savoir scientifique*). Margolinas (2012) tells the difference between knowing (*saber*) and knowledge (*conhecimento*), which is a fundamental point of the Didactics of Mathematics. Thus, she reports that knowing is the balance reached between the individual and the application site, including a series of factors, such as: the knowledge of the body, the knowledge of action, of interaction, among others. Knowledge, on the other hand, is considered as a social and cultural construction, being then decontextualized, formalized, memorized, corresponding to the textual nature. Thus, in Figure 2, we have the process of returning knowing and knowledge, which is not trivially considered during the intention to teach. With this, the teacher - researcher must propose to students the teaching situations, allowing them to seek knowledge corresponding to the knowing (*le savoir mathematique*) that already exists.

Figure 2

Modifications involving the mathematical knowing (le savoir) and mathematical knowledge (connaissance) provided by the TDS. (Margolinas, 2012, p. 8)



However, it is also noteworthy that the TDS serves as a basis for the conception and proposition of the problem situations that will be elaborated, forming a didactic accomplishment, and stimulating the students' knowledge and the construction of knowledge. This process will take place in a invisible way, where the answers are not easily obtained. Finally, it is worth mentioning that it is necessary to compare the results and classroom conditions from which the data were collected, aiming at an eventual replication of the research. With this, there is the possibility of replicating the work for the area of Mathematics teaching, according to an action planned by the teacher (Alves, 2016).

Then, an epistemic-mathematical field of interest is initiated and demarcated relating to the Padovan sequence, with emphasis on the generalization process of these numbers. Some theorems and properties and a brief history of this numerical sequence are presented for later elaboration of teaching situations.

EPISTEMIC-MATHEMATICAL FIELD

Aiming to investigate the process of generalization of the Padovan sequence emphasizing the generalization of the initial terms and their coefficients, the researchers involved carried out studies based on bibliographic surveys. The study considered a relation of the Fibonacci sequence, the Padovan sequence, is a linear and recurring sequence of integers, named after the Italian architect Richard Padovan, born in the city of Padua in 1935 (Stewart, 1996). The French mathematician Gerard Cordonnier (1907-1977) also contributed to the study of this sequence, discovering the radiant number or plastic number. With that, this sequence is also known as Cordonnier sequence, having its recurrence formua $P_n = P_{n-2} + P_{n-3}, n \geq 3, \forall n \in \mathbb{N}$ and with $P_0 = P_1 = P_2 = 1$. Its characteristic polynomial is given by the following equation $x^3 - x - 1 = 0$, with three roots, one real and two complex and conjugated.

Unlike the Fibonacci numbers, where it has its genesis in the problematic reproduction of immortal rabbits (Singh, 1985; Stakov, 2009), Padovan numbers do not have an initial problem situation. However, during its process of historical and mathematical development, an evolution of this sequence can be perceived, with forms of matrix representations of its generalization emerging, some registered in recent works and others not yet registered.

In Spinadel and Buitrago' (2009) work, a study on the convergence between neighboring terms and their relationship with the plastic number is carried out. Sokhuma (2013) and Seenukul et al. (2015), developed some properties of the matrix form of the Padovan sequence. In turn, Vieira and Alves (2018) define a generalization for the initial values of the sequence, calling it Padovan-Like sequence. Finally, Gulec and Taskara (2012) and Turkmen (2008) generalize the coefficients of another linear and recurring sequence, but in this work, the same analytical reasoning for Padovan sequence in its matrix form, is then presented.

Property 1. The convergence relationship between Padovan's neighboring terms is (Spinadel & Buitrago, 2009):

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} \approx 1,32 = \psi$$

Theorem 1. The Binet formula of the Padovan sequence is given by:

$$P(n) = A(x_1)^n + B(x_2)^n + C(x_3)^n,$$

where $n \in \square, x_1, x_2, x_3$ are the roots of the characteristic polynomial and $A = \frac{(x_2 - 1)(x_3 - 1)}{(x_1 - x_2)(x_1 - x_3)}, B = \frac{(x_1 - 1)(x_3 - 1)}{(x_2 - x_1)(x_2 - x_3)}, C = \frac{(x_1 - 1)(x_2 - 1)}{(x_3 - x_1)(x_3 - x_2)}$ their respective coefficients.

Theorem 2. The matrix generating the Padovan sequence is given by (Sokhuma, 2013; Seenukul et al., 2015):

$$\text{For } Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ we have } Q^n = \begin{bmatrix} P_{n-2} & P_{n-3} & P_{n-4} \\ P_{n-1} & P_{n-2} & P_{n-3} \\ P_{n-3} & P_{n-4} & P_{n-5} \end{bmatrix}, \text{ for every } n \geq 5.$$

Definition 1. The Padovan-Like sequence is defined by (Vieira & Alves, 2018):

$$P_n = P_{n-2} + P_{n-3},$$

$n \geq 3, \forall n \in \square$ and with $P_0 = a_0, P_1 = a_1, P_2 = a_2$ in which $a_0, a_1, a_2 \in \square$.

Theorem 3. The generating matrix of the Padovan-Like sequence, given the initialization vector *w* and $n \in \square$, is given by (Vieira & Alves, 2018):

$$wQ^n = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} P_{n+2} & P_{n+1} & P_n \end{bmatrix}, n > 0.$$

Theorem 4. Binet formula for the Padovan-Like sequence is given by (Vieira & Alves, 2018):

$$P(n) = A(x_1)^n + B(x_2)^n + C(x_3)^n,$$

where $n \in \square, x_1, x_2, x_3$ are the roots of the characteristic polynomial and $A = \frac{a_2 - a_1 x_2 - a_1 x_3 + a_0 x_2 x_3}{(x_1 - x_2)(x_1 - x_3)}, B = \frac{a_2 - a_1 x_1 - a_1 x_3 + a_0 x_1 x_3}{(x_2 - x_1)(x_2 - x_3)}, C = \frac{a_2 - a_1 x_1 - a_1 x_2 + a_0 x_1 x_2}{(x_3 - x_1)(x_3 - x_2)}$ their respective coefficients.

Definition 2. For $S_1, S_2 \in \square$, the Padovan sequence (S_1, S_2), represented by $P_n(S_1, S_2)$ with $n \geq 3$ and $n \in \square$, is defined by the recurrence:

$$P_{n(s_1,s_2)} = s_1 P_{n-2(s_1,s_2)} + s_2 P_{n-3(s_1,s_2)},$$

with the initial values $P_{0(s_1,s_2)} = P_{1(s_1,s_2)} = P_{2(s_1,s_2)} = 1$. For notation purposes, $P_{n(s_1,s_2)} = B_n$ is used

Theorem 5. The matrix generating the Padovan sequence (S_1, S_2), for $n \geq 1$ and $n \in \square$, is given by:

$$vQ^n = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ s_1 & 0 & 1 \\ s_2 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} B_{n+2} & B_{n+1} & B_n \end{bmatrix},$$

where v represents the line vector with the initial values of the Padovan sequence.

To carry out a didactic transposition of these mathematical contents, some properties to be explored in teaching situations were selected, providing students with an understanding of these relationships, and investigating their intuitive thinking. Therefore, the proposed didactic situations are analyzed and discussed based on the TDS.

DESIGN OF THE DIDACTIC SITUATIONS

The conception of the didactic situation present in this work is a process of generalization of the Padovan sequence, transforming it into content to be taught in class, since the epistemic-mathematical field developed in this research is discussed only in courses of pure mathematics. However, to investigate and explore the theorems and mathematical properties, a didactic hypothesis regarding the mathematical object under study is elaborated, based on the DE with a focus on the TDS, isolating specific notions and mathematical properties, transposing this object to an existing context in the classroom. This fact is defined, according to epistemologists, as didactic transposition (Brousseau, 1997; 2002).

Thus, when designing didactic situations, either microdidactic or macrodidactic variables can be are defined. The first is to predict student behavior and possible obstacles that may be encountered during the didactic situation. The second, the research choices are directed according to the general structure of the DE (Santos & Alves, 2017). Thus, this research uses the microdidactic variable, so there is a relationship between the mathematical content referring to the sequence under study and the proposed problem situations. A problem situation consists of a discursive question, with clear and objective statements that uses the mathematical objective implicit, so that it is achieved during its resolution (Almouloud, 2016). The choice of theorems and properties requires students' little mathematical knowledge since their demonstrations are carried out through inductive steps.

A PRIORI ANALYSIS OF THE DIDACTIC SITUATIONS

To predict students' potential behavior in each of the phases foreseen by the TDS, a descriptive and predictive action was carried out, raising some didactic hypotheses according to the problem situations, being, therefore, characterized as a preliminary analysis. In this sense, the students' cognitive side is then stimulated, so that theoretical mathematical knowledge can be developed, given the phases of action, formulation and validation based on the teaching methodology of the TDS (Oliveira & Alves, 2019).

To explore the process of generalization of the Padovan sequence, three problem situations were elaborated according to the definitions and properties studied in the epistemic-mathematical field. With that, the work will investigate that process around these numbers originated from the recurrence, relating them to other elementary mathematical contents (matrices, systems, equations, etc.) linked to the activity of the Math teacher, aiming to elaborate an action plan to attend the objectives assumed in the research.

Problem situation 1: Following the same perspective of evolution of the mathematical content, we will develop studies addressing new properties of sequences with similar characteristics to the Padovan sequence. Thus, called by Vieira and Alves (2018) Padovan-Like sequence, we have the following terms of this sequence: a_0, a_1, a_2, $a_0 + a_1$, $a_1 + a_2$, ... Using the base generating matrix of Seenukul et al. (2015) and Sokhuma (2013), how can we obtain the terms of this sequence?

In this first problem situation, at the time of the action, the recurrence formula of this sequence must be found so, later, its matrix form will be obtained. During formulation, students must keep in mind the generating matrix of Seenukul et al. (2015) and Sokhuma (2013), and its construction rule, thus creating an initialization vector containing these initial values. Thus, this vector must be multiplied to the left of the generating matrix, building Theorem 3, $wQ^n = \begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^n = \begin{bmatrix} P_{n+2} & P_{n+1} & P_n \end{bmatrix}, n > 0$. In the validation, the demonstration must be carried out through the principle of finite induction, verifying the generating matrix.

In the last phase, institutionalization, the teacher resumes the didactic situation and checks the students' resolutions, discussing the importance of using the generating matrix for the Padovan-Like sequence, where its initial terms are generalized.

Problem situation 2: Can we then write a function in which it is possible to find the terms of that sequence, called Padovan-Like sequence, without the need to know its previous terms? (Binet formula).

During the resolution of this problem situation, in action, students should use the general Binet formula, in which $f(n) = Ax_1^n + Bx_2^n + \ldots + Zx_z^n$, where $A, B, \ldots, Z$ are the coefficients of the Binet formula and $x_1, x_2, \ldots, x_z$ are the roots of the characteristic polynomial of the sequence. The formulation starts with the construction of the system of linear equations, by inserting the initial values of this sequence, the following terms $P_0 = a_0$, $P_1 = a_1$, $P_2 = a_2$.

In the validation phase, they must solve the system of equations and obtained the formula $P(n) = Ax_1^n + Bx_2^n + Cx_3^n$ with $A = \frac{a_2 - a_1x_2 - a_1x_3 + a_0x_2x_3}{(x_1 - x_2)(x_1 - x_3)}$, $B = \frac{a_2 - a_1x_1 - a_1x_3 + a_0x_1x_3}{(x_2 - x_1)(x_2 - x_3)}$,

$C = \frac{a_2 - a_1x_1 - a_1x_2 + a_0x_1x_2}{(x_3 - x_1)(x_3 - x_2)}$ with x_1, x_2, x_3 being the roots of the characteristic equation, and $n \in \mathbb{N}$. Finishing the last phase of the TDS, institutionalization, the teacher formalizes this situation, reporting the existence of the representation of this formula for the Padovan-Like sequence.

Problem situation 3: Based on works by Gulec and Taskara (2012) and Civciv and Turkmen (2008) and, according to Table 1, the terms of the sequence (S_1, S_2) - Padovan are represented. Given that, check if there is any relationship between this table and the terms (1,1,1,2,2,3,4,...). If it exists, explain it in detail and determine some terms of that sequence.

Table 1

Terms of the sequence (S_1, S_2) *-Padovan.*

N	$B_n(s_1,s_2)$
0	1
1	1
2	1
3	s_1+s_2
4	s_1+s_2
5	$s_1(s_1+s_2)+s_2$

Is it possible to obtain a generator matrix for this sequence studied? If so, demonstrate it.

According to Table 1, during the action phase students should relate these numbers to the Padovan sequence, establishing the relationship $B_n = s_1B_{n-2} + s_2B_{n-3}, n \geq 3$, with $s_1, s_2 \in \mathbb{R}$ and presenting the initial values $B_0 = B_1 = B_2 = 1$, thus generalizing the coefficients of the Padovan sequence recurrence formula. Continuing this generalization process, they should, in formulation, obtain the generating matrix based on the matrices previously studied by Seenukul et al. (2015) and Sokhuma (2013), and the Padovan-Like sequence matrix (Problem situation 1). However, the operator is constructed in the first column of the matrix, bringing with it the coefficients of the sequence recurrence formula. The remaining rows and columns remain similar to the Padovan matrix studied. It is necessary to insert a vector with the initialization values of the sequence, which is multiplied to the left of the matrix, comparably to the problem situation 1. At the end of this phase, students should obtain the following matrix $\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ s_1 & 0 & 1\\ s_2 & 0 & 0\end{bmatrix} = \begin{bmatrix}B_3 & B_2 & B_1\end{bmatrix}$, which, when raised to the n-th power, results in $\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ s_1 & 0 & 1\\ s_2 & 0 & 0\end{bmatrix}^n = \begin{bmatrix}B_{n+2} & B_{n+1} & B_n\end{bmatrix}$.

In the validation, the generating matrix found must be demonstrated by mathematical induction, thus validating the mathematical theorem formulated in this activity. Finally, in institutionalization, the teacher explains the construction of a matrix that generates a linear and recurring sequence and the initialization vector. Facing a perspective of knowledge evolution, according to the TDS, a model of generalization of the Padovan sequence is shown, stemming from the students' understanding of the contents covered.

EXPERIMENTATION

In the third phase of the ED, experimentation, according to Lopes, Palma and Sá (2018):

> It initially consists of the period of application and experimentation of the activities previously planned, collecting data on the investigation. In a second step, it refers to the analysis of the results that will be obtained in the investigation. This phase is based on the analysis of all the data obtained in the experimentation during the teaching sessions, as well as productions inside or outside the classroom. (Lopes, Palma & Sá, 2018, p. 164).

It is at this moment that the application of the activities elaborated occurs, but the DE lets the way in which this data will be collected and analyzed free, and the need to use a teaching methodology is felt. After studies based on the work of Brousseau (2008, 1986, 1976), the TDS was selected as the teaching methodology for this research because it instigates the student's intuitive side.

This phase took place at the Federal Institute of Education, Science and Technology of Ceará (IFCE), *campus* of Fortaleza, in 2019, during the Mathematics History course of the Pre-Service Mathematics Teacher Training Course, encompassing eight students (teachers) who were enrolled in the discipline, who accepted participating in this research. The application took place through three problem situations, which were discussed and resolved in groups that elaborated resolution strategies, therefore, describing them on papers and/or a whiteboard. The elaboration and analysis of these activities were based on the TDS, through photographic and audio recordings during synchronous productions and activities of the subjects (teachers in initial training).

A POSTERIORI ANALYSIS AND INTERNAL VALIDATION

During the application, Almouloud (2007) states that there may be a need for possible corrections and adjustments, considering the most relevant elements during the experimentation process. After that, a posteriori analysis takes place, evaluating and analyzing the results obtained, so that there is a contribution of didactic knowledge during the transmission of the content. However, concluding this analysis, there is then the validation of the elements in the experimentation phase of the DE, by comparing it with the results discussed by the students, besides arguing over the evolution or not of engineering in this research. The problem situations that will be analyzed below are examined in the light of a microdidactic variable, since they refer to the organization of an experimentation phase, as predicted by a DE (Almouloud, 2016).

With the development of didactic situations, in view of the proposed activities, this research was validated. Aiming to encourage the students' investigative side, they were induced to solve the problems presented, through a didactic contract. The three problem situations enabled students to understand the contents related to the generalization process of the Padovan sequence, according to the didactic variables. These variables are then selected according to the way the questions are posed.

Starting this process, we have the matrix form of the Padovan-Like sequence generalizing the initial terms of the sequence. Following the reasoning, the closed formula for Padovan-Like's terms, known as Binet formula, is analyzed. Finally, the matrix form of the sequence is studied, with the generalization of the coefficients of the recurrence formula. During the validation phase of the TDS, the inductive method was used to validate the theorems and properties, as it is a demonstration method known to the group of students involved.

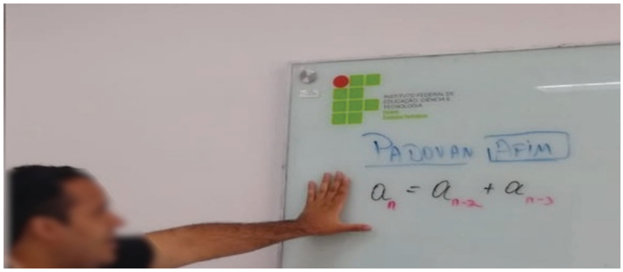
Starting the data application, we distributed a list of exercises to the students, and advised them to interact with each other. Those activities were examined and discussed during resolution, when students formulated arguments aiming at their further examination, which was registered through video-recordings and photos (Oliveira, 2018).

In problem situation 1, the general objective was to obtain the matrix form of the Padovan-Like sequence, from the Padovan sequence generating matrix defined by Seenukul et al. (2015) and Sokhuma (2013), offering students a mathematical basis, so that they could solve the activity. During data collection, some difficulties to create the vector with the initial terms of the sequence were observed. However, with the help of the other colleagues during the discussions, they could reason and resolve the issue.

In Figure 3, we can notice the action taking place, when the students understand that the recurrence formula will be the same, with different initial terms only, therefore: $P_n = P_{n-2} + P_{n-3}, P_0 = a_0, P_1 = a_1, P_2 = a_2$ e $n \geq 3, n \in \mathbb{N}$. We observe in Figure 3 that Student A named the n-th term of the Padovan-Like sequence a_n, with no prejudice to the development of his resolution.

Figure 3

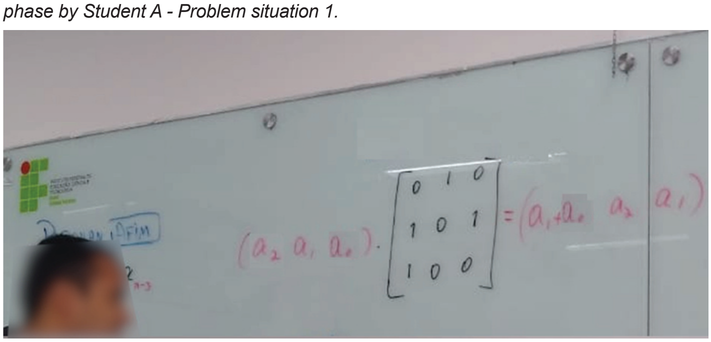
Action phase by Student A - Problem situation 1.



During the formulation phase, we noticed that Student A managed to build a vector, containing the initial values of the sequence, from the recurrence formula verified in the previous phase. By multiplying, he obtained the matrix generating the Padovan-Like numbers, as shown in Figure 4. Adding the first element of this matrix and applying the recurrence, the student concluded that $a_1 + a_0 = a_3$, simplifying the result. The student only found it difficult in this phase to construct the initialization vector, which was remedied by his interaction with the other colleagues on definitions and concepts of linear algebra.

Figure 4

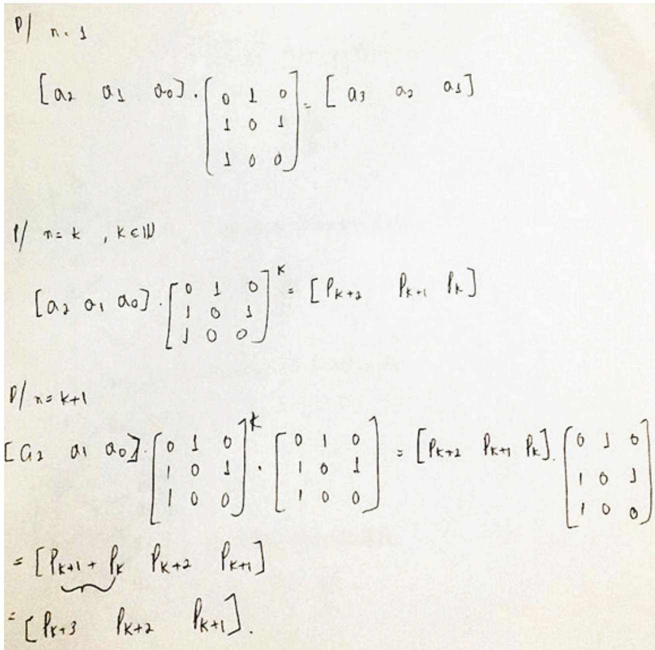
Formulation phase by Student A - Problem situation 1.



To validate this matrix, all students participated in this process, performing the mathematical inductive step to demonstrate Padovan-Like's generating matrix. In this sense, Figure 5 shows this phase performed by Student C, being then selected due to mathematical and written organization issues. We noticed that this student simplified the matrix of order 3x3, and replaced the n-th of the sequence with P_n. Below, we indicate elements of the student's resolution.

Figure 5

Validation phase by Student C - Problem situation 1. (Research data)



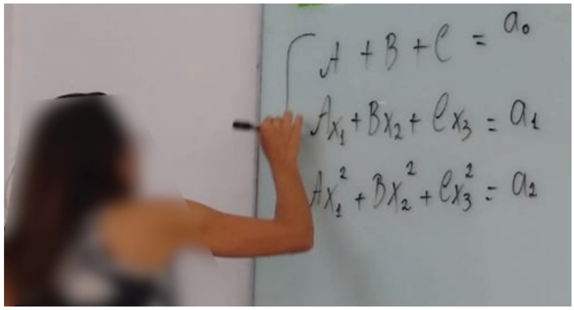
After that, we had the resolution of problem situation 2, in which students were given Binet variant formula, where, based on the Padovan sequence, they should obtain this formula for the Padovan-Like numbers. Based on this, Student B commented:

> Student B: [...] this formula will be basically the same as Padovan's, and when solving the system of linear equations, we must enter the generalized initial values to obtain the new Binet formula of these numbers similar to Padovan's [...].

This statement shows the students have some knowledge of the Binet variant formula, starting the action phase, since this discussion served as a basis for them to assemble the system of equations. Figure 6 presents Student F carrying out the formulation phase, describing this system with three equations and three unknowns, inserting the generalized initial values.

Figure 6

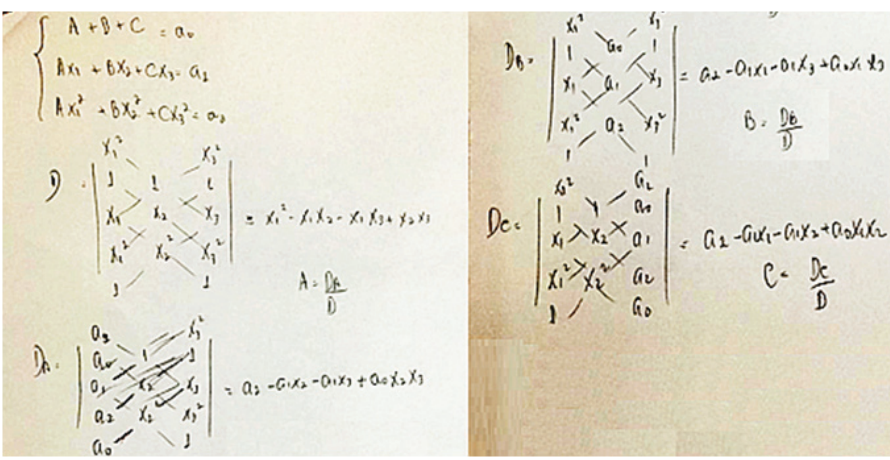
Formulation phase by Student F - Problem situation 2.



The student found some difficulties to solve the system of equations with three variables; however, with the participation and assistance of the other students, she reached the resolution, as shown in Figure 7, during a validation phase. Student E calculated the coefficients of the system using Cramer's rule, the roots of the characteristic equation were kept with their respective nomenclatures (x_1, x_2, x_3), and their values were not used to calculate the coefficients during resolution. The students were aware that to calculate a term in the sequence using the Binet formula, it is necessary to know the values of the roots of the equation.

Figure 7

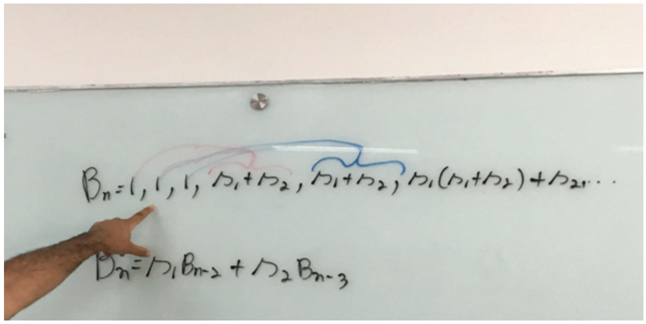
Validation phase by Student E - Problem situation 2. (Research data)



Finally, analyzing the problem situation 3, in which the coefficients of the sequence were generalized, we can identify the action phase in Figure 8. Student D could establish the recurrence formula for this sequence, comparing it with the Padovan sequence, thus realizing the insertion of two variables, s_1, s_2, in the coefficients.

Figure 8

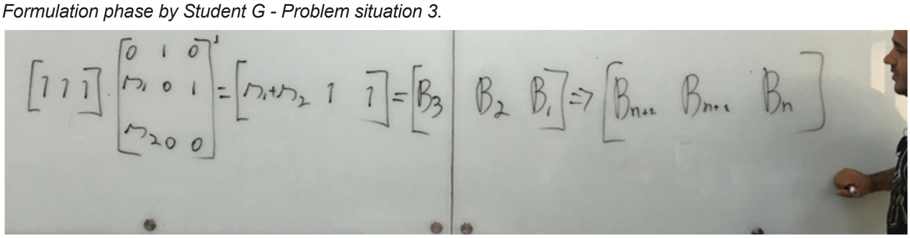
Action phase by Student D - Problem situation 3.



In the formulation phase, the student obtained the matrix form of this sequence, based on Padovan's generating matrix, and on the initialization vector studied in the first problem situation. Student G could understand that the first column of the generating matrix carries with it the values of the coefficients of the recurrence formula, as shown in Figure 9.

Figure 9

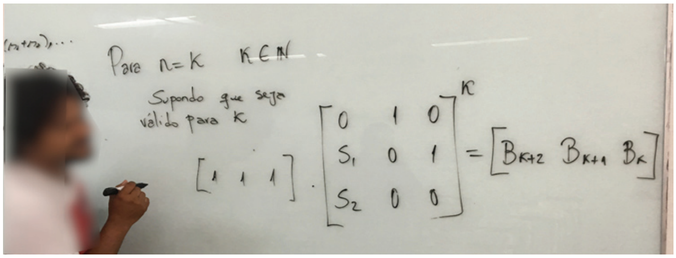
Formulation phase by Student G - Problem situation 3.



In Figure 10, Student H started the matrix validation process, through the mathematical inductive step, the first step being performed for $n = 1$, when Student G obtained a line matrix, after using the inductive model (see Figure 9).

Figure 10

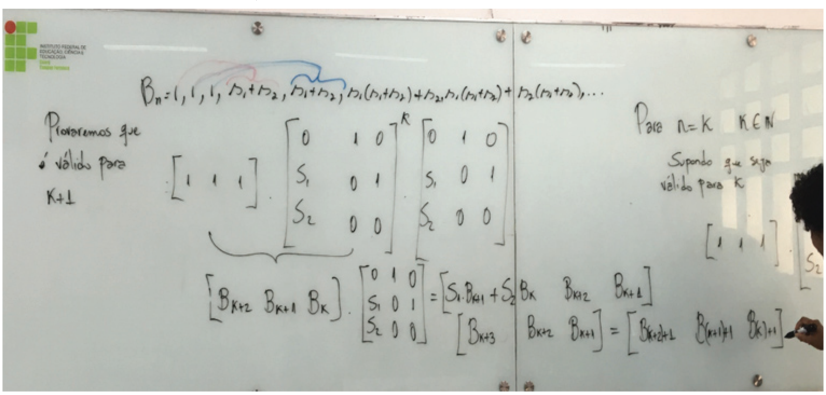
Validation phase by Student H - Problem situation 3.



Finishing this phase, Student H, in Figure 11, continued the validation process, checking the validity for $n = k + 1$. With this, it is possible to validate a theorem not found, until then, in the literature and scientific articles about pure mathematics, referring to Padovan numbers.

Figure 11

Finalization of the validation phase by Student H - Problem situation 3.



The teacher-researcher carried out the institutionalization at the end of each resolution of the problem situations, analyzing the students' productions and highlighting the evolutionary process of the sequence around its generalization. In the first problem situation, the student could validate Theorem 3, obtaining the matrix form of the Padovan-Like sequence, seen in the work of Vieira and Alves (2018). Concerning these authors' work, in problem situation 2, Theorem 4, starting from Theorem 1, could be demonstrated through the examination of the concepts referring to Binet variant formula (Alves, 2015; 2017). In the third and last problem situation, a new content was inserted related to the generalization of the coefficients of the recurrence formula of the Padovan sequence, which demonstrated Theorem 5.

The validation of this research took place internally (Laborde, 1997) since only the productions observed in this work and the group of teachers involved were analyzed, not being compared with other external productions and, also because the number of participants is considered small (eight teachers in initial training). In this process, the phases of *a priori* and *a posteriori* analysis were compared. In the first, the epistemic-mathematical field is introduced, with some mathematical properties and theorems. In the second analysis, some formal theorems are selected, and problem situations are elaborated. Then, the students' resolutions are analyzed during the experimentation phase of the DE.

The students expressed great interest in the research, as evidenced in their comments, a feeling of discovery of properties, such as:

> Student A: [...] I felt I was building my own knowledge, especially because I had never seen this sequence before, not in mathematics, let alone in the way that is being passed on to us [...]. Some of these issues are not found in the literature, this shows we are more interested in the resolution of the exercises [...]

The institutionalization phase of the TDS is important in this validation process of this research because, from an epistemological perspective, it is at this moment that the verification and construction of the mathematical concepts addressed in the classroom takes place. It was possible to perceive that the students (teachers in initial formation) managed to build the process of generalization of the Padovan sequence, investigating mathematical theorems and discovering new definitions, besides considering its cognitive and didactic - methodological aspects.

Some epistemological and cognitive obstacles were encountered during the resolutions, and some difficulties or stages of inertia or stagnation in the formulation of some mathematical properties and their corresponding validation were observed, since some students (teachers in initial training) argued not to remember the principle of the finite induction and the notion of vectors. However, based on the exchange of information and the scientific debate with the other participants, these obstacles were resolved.

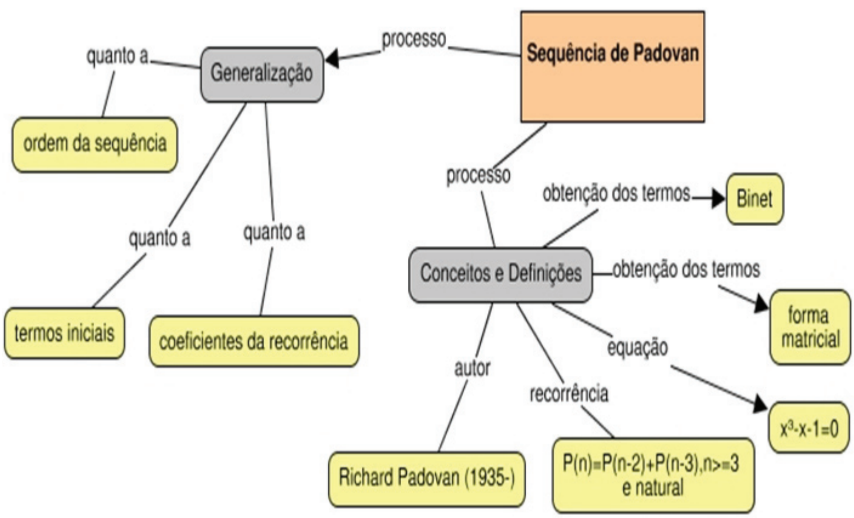
Thus, in the didactic field, through didactic situations, we could understand the process of collective construction through the set of theorems selected and presented, encouraging the students' inductive reasoning and their understanding in the evolutionary epistemological process on the Padovan sequence. From the cognition point of view, the students' mathematical evolution was observed in the classic phases of action, formulation and validation provided by the TDS.

CONCLUSIONS

Based on the assumptions of a DE, all internal analyses were carried out in this research, verifying the three activities (structured didactic situations) proposed during the experimentation phase and based on the TDS, assisting the teacher's practice in the area of Mathematics and the History of Mathematics in initial teacher training courses. The importance of the French influence in didactic culture is emphasized, since it was built based on the constructivist theory of knowledge (Artigue, 1995). In view of the investigations developed in the classroom, structured didactic situations that can be applied or replicated in other experiments were outlined.

In the first phase, an extensive bibliographic survey was carried out around the DE, the TDS and the content referring to the generalization process of the Padovan sequence. We selected this object of study because of the few works related to the teaching of this sequence, addressing only the mathematical part in pure mathematics courses. Figure 12 shows the conceptual map of this sequence of investigation, with contents to be select and approached in undergraduate courses, given its evolutionary process. Thus, we can visualize the process of mathematical generalization, mathematical induction and discovery of concepts and new basic definitions, essential for the study of these numbers (see figure 12).

Figure 12
Conceptual map of the Padovan sequence.



During the experimentation phase foreseen by the DE, students (teachers in initial training) were able to perceive the unimpaired epistemological and evolutionary process in face of the possibility of generalizing this numerical sequence, expressing their amazement when determining mathematical theorems and formulating new mathematical definitions over the course of situations, up until now not studied nor found in the scientific literature (Vieira & Alves, 2018; 2019). Despite the difficulties or obstacles found to solve the problem situations, the general objective of this work could be achieved. This work is an excerpt from research carried out in the Graduate course in Science and Mathematics Teaching at IFCE, approved by the Research Ethics Committee (CEP) (review: 3.314.835), since the study used a representative set of empirical data carried out with the participation of human beings.

Furthermore, the specific objectives were achieved, when we could carry out a didactic transposition (Chevallard, 1991) relating to the mathematical content discussed, thus fostering the students' understanding of the evolutionary and epistemological process of this sequence, which is still little discussed by authors of History of Mathematics books. Finally, we acknowledge the importance of these activities in the context of the History of Mathematics, also involving mathematics teachers training (Alves, 2015; 2017) and other possibilities of using the technology. With the use of the DE and the TDS, we could recognize the didactic situations created, which can also be reproduced in other places of experimentation and investigation, supported by the academic *locus*.

AUTHORS' CONTRIBUTION STATEMENTS

F.R.V.A. supervised the project. F.R.V.A., P.M.M.C.C. and R.P.M.V. conceived the idea presented and discussed in this research. R.P.M.V. developed the theory, adapted the methodology to the context of the experienced classroom, creating models and collecting the data. F.R.V.A., P.M.M.C.C. and R.P.M.V analyzed the data and discussed to write the final contribution of the manuscript.

DATA AVAILABILITY STATEMENT

The authors agree that the data supporting the results of this study are available upon reasonable request, at the discretion of the authors.

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