

Primary School Teacher's Professional Learning: Exploring Different Meanings of the Equals Signal

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Received for publication on 19 Sep. 2020. Accepted after review on 11 Jul. 2020

Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Context: The use of professional learning tasks in practicing teacher education programmer has become a useful way to mobilize and construct knowledge to teach mathematics. **Objective:** The paper has the aim to identify as professional learning tasks based on classroom practice contribute to the mobilization and expansion of the algebraic thinking of primary school teachers, regarding to the different meanings of the equal sign. **Design:** This research is part of a qualitative methodology and a theoretical interpretive perspective. **Settings and Participants:** The research had been realized in a formative process composed of 14 meetings held in a municipal public school in São Paulo, with the participation of 6 teachers from the early years. **Data collection and Analysis:** Data were collected by documents and observation with video and audio recording, and analyses were done under a deductive approach. **Results:** Reflections and discussions that occurred during the meetings allowed us to identify traces of signification and construction of the teachers' algebraic thinking. **Conclusion:** The study made it possible to identify resignification by the teachers of the sign of equality, going from operational to equivalence and relational.

Keywords: Practicing Teacher Education; Algebraic Thinking; Teacher Knowledge; Equal Sign; Professional Learning.

Aprendizagem Profissional de Professores dos Anos Iniciais: Explorando os Diferentes Significados do Sinal de Igualdade

RESUMO

Contexto: A utilização de tarefas de aprendizagem profissional em processos de formação continuada tem se tornado um profícuo caminho para a mobilização e a construção de conhecimentos para ensinar matemática. **Objetivo:** O artigo tem por objetivo identificar como tarefas de aprendizagem profissional, fundamentadas na prática letiva, contribuem para a mobilização e a ampliação do pensamento algébrico de professores dos anos iniciais, no que se refere aos diferentes

significados do sinal de igualdade. **Design:** Esta pesquisa¹ insere-se em uma metodologia de cunho qualitativo e em uma perspectiva teórica interpretativa. **Ambiente e participantes:** A pesquisa se deu em um processo formativo composto por 14 encontros realizados em uma escola pública municipal de São Paulo, com a participação de 6 professoras dos anos iniciais. **Coleta e análise dos dados:** Os dados foram recolhidos por meio de documentos e observação com gravação em áudio e vídeo e as análises foram realizadas por uma abordagem dedutiva. **Resultados:** As reflexões e as discussões que ocorreram no decorrer do percurso dos encontros permitiram identificar indícios de significação e construção do pensamento algébrico das professoras. **Conclusões:** O estudo permitiu identificar ressignificação por parte das professoras do sinal de igualdade, passando de operacional para equivalência e relacional.

Palavras-chave: Formação Continuada; Pensamento Algébrico; Conhecimento do Professor; Sinal de Igualdade; Aprendizagem Profissional.

INTRODUCTION

The continuing education of teachers who teach mathematics and who pursue learning during their practices is a vast field of research (Opfer & Pedder, 2011; Ponte et al., 2008; Serrazina, 2013; Webster-Wright, 2009), and this study is part of this questioning. The possibilities of building specific mathematical and pedagogical knowledge need to be explored in mathematics teaching (Ball, Thames & Phelps, 2008), and one of the major paths is precisely through classroom situations that can contribute to a reflective context (Silver et al., 2007; Smith, 2001).

Rethinking continuing education of teachers who teach mathematics remains a crucial theme (Fiorentini & Crecci, 2017), for which the professional learning tasks (PLT) seem to be a potential means to both invigorate the changes in the practice of primary school teachers and enable (new) professional learning (Ball & Cohen, 1999).

Also, developing students' algebraic thinking (AT) since their early years of education is fundamental to open doors to the algebra field in the higher grades (Blanton & Kaput, 2008; Kieran et al., 2016). The international literature has been demonstrating how relevant it is to develop AT since the initial years of schooling (Canavaro, 2007; Kieran et al., 2016; Molina, 2011). Other international researches, in the same trend of the AT development, states that teachers must mobilize specific knowledge and (re) structure their practice, so that they can develop AT in their classrooms (Britt & Irwin, 2011; Kieran et al., 2016; Ponte & Branco, 2013).

Currently in Brazil, concerning AT, we find the "Algebra" axis for the initial years included in the National Common Curricular Base (BNCC) (Brazil, 2017) and in the São Paulo *City Curriculum* (PMSB, 2017). The documents recommend that educators should propose patterns and sequences in the 1st and 2nd grades and, later, as of the 3rd grade, properties of the concept of equality and the different meanings of the equals sign.

¹ The research is part of the project "Conhecimento matemático para o ensino de álgebra: uma abordagem baseada em perfis conceituais", approved by the Research Ethics Committee of UFABC under number CAAE 55590116.8.0000.5594.

Therefore, taking national and international literature and curricula guidelines on AT development in the early years of schooling, in our research, we decided to focus on an approach that explores the different meanings of the equality sign, or equals sign (Kieran, 1981; Ponte, Branco & Matos, 2009; Trivilin & Ribeiro, 2015). Preponderant elements are identified in the triggering of our driving question: to what extent professional learning tasks grounded in practice foster the mobilization and construction of teachers' professional knowledge to develop students' AT in the early years of education? Based on this problem, we aimed to identify how professional learning tasks (PLT) based on teaching practice contribute to the mobilization and expansion of the algebraic thinking (AT) of primary school teachers by involving the different meanings of the equals sign .

To respond to this objective, we divided this article into seven sections. In the first three sections, we address the different thematic axes that make up the literature review, to approach the problem identified and the main theoretical elements of the study. Then, we present the research context and the structural methodological aspects, and the last three sections present the events and the development of the data analysis. The last section brings the conclusions and final considerations.

TEACHERS' CONTINUING EDUCATION AND PROFESSIONAL KNOWLEDGE

To start this discussion, we report to Fiorentini and Crecci (2017), who highlight the importance of continuing education and research as an open field in the Brazilian context. The authors highlight the gaps in literature, discussing the absence of a theoretical and methodological framework that helps researchers to analyze "how teachers' professional knowledge and skills² are problematized, woven in school practice" (Fiorentini & Crecci, 2017, p. 181).

Advocating for continuing education for teachers who teach mathematics in the early years, it is paramount to understand how continuing education based on reflections can highlight the importance of building specific knowledge to teach mathematical concepts (Ponte et al., 2008). Also, it seems central to consider building an environment that is favorable to discussions and task analysis (Ball & Cohen, 1999; Silver et al., 2007; Smith, 2001). In the context of continuing education, in our study, we used Serrazina model (2013).

Supporting the development of collective professional knowledge more broadly and elucidating appropriate ways to capture and communicate this type of knowledge usefully are considered essential to foster professional learning (Ball, Ben-Peretz, & Cohen, 2014). And to complement this perspective, collective discussions, indicated as one of the bases of teachers' learning, are included, as they allow exchanges with other professionals in the

² Although there is a discussion in the literature about knowledge and skills, in this article, due to conceptual rapport, we use professional knowledge as defined by Shulman (1986).

field, so that they can understand, compare, (re)formulate their (un)certainties, expanding their opportunities to learn (Ball & Cohen, 1999; Ribeiro & Ponte, 2019).

The idea that we can only teach what we know, and that the ultimate test to confirm whether we have actually understood something is our ability to teach, transforming knowledge itself into a teaching and learning possibility (Shulman, 1986, 1987), converge with the opportunities to (re)formulate (un)certainties and expand the possibilities of learning, as previously addressed by other authors. Hence, one of our theoretical framework bases is Ball, Thames and Phelps's (2008) work, developed from the studies of Shulman (1986) on the notion of *Mathematical Knowledge for Teaching* (MKT).

The MKT theoretical model starts from the assumption of how needed and important it is to understand both the knowledge teachers need to address in their teaching practice, and how it is possible to mobilize that knowledge in the classroom. Therefore, Ball, Thames and Phelps (2008) suggest a mathematical knowledge basis to support the accomplishment of the teaching tasks. The MKT is based on the Specific Content Knowledge (CK) and refers to the mathematical content to be taught, subdivided into: Common Content Knowledge (CCK), the knowledge used for purposes other than teaching and which are usually found and used by people from the most diverse professions; Specialized Content Knowledge (SCK), a type of knowledge aimed at the teacher and his craft of teaching mathematics; Horizon Content Knowledge (HCK), the kind of mathematical knowledge that enables the teacher to understand the disposition of the mathematical concepts throughout the curriculum and know how to connect and revisit them whenever possible and necessary.

Besides the domains already explored, Pedagogical Content Knowledge (PCK) is brought up, constituted from an amalgam between the specific content and the general pedagogical content, which are united in such a way that they end up "becoming" a type of knowledge that enables and fosters the teaching of mathematics, for example. The PCK is also subdivided in the work of Ball, Thames, and Phelps (2008) into three domains: Knowledge of the Content and Students (KCS), which, for example, allows the teacher to anticipate mistakes students may make; Knowledge of Content and Teaching (KCT), which allows teachers to establish relationships about teaching and mathematics, enabling them to know how to choose the best examples and how to sequence them to overcome students' mistakes or difficulties; Knowledge of the Content and Curriculum (KCC), the knowledge about how the content is distributed throughout the curriculum that will be studied/taught.

It is known that teachers' knowledge is a fundamental aspect of their education, since it is interrelated with the teachers' level of confidence, both in terms of mathematics and teaching, or in relation to what they consider their students can learn in mathematics (Serrazina, 2013). Hence, there is a positive relationship between the teachers' confidence and the improvement of their mathematical knowledge (Serrazina, 2013).

ALGEBRAIC THINKING AND DIFFERENT MEANINGS OF THE EQUALS SIGN

With regard to the teaching of algebraic thinking, we begin by discussing how important it is to understand that mathematics is a science that works with patterns (Ponte, Branco, & Matos, 2009) and is present in the daily lives of countless professions and in everyday life. Thus, we can infer that learning mathematics permeates school domains and directly impacts students' future relationships (Skovsmose, 2005). The fact that mathematics fosters regularity refers to the observation and definition of patterns and generalization, thus finding some of the main contributions that characterize the teaching of AT, especially in the early school years (Britt & Irwin, 2011; Kieran et al. 2016). Based on the assumption that there is no single definition for AT (Ribeiro & Cury, 2015), this article is grounded on the notion that AT is "a habit of the mind that permeates all mathematics and that involves the students' ability to build, justify, and express conjectures about mathematical relationships and structures" (Blanton & Kaput, 2008, p. 142).

Algebra teaching since the early years aims to encourage a way of thinking, that is, a habit of seeking regularities and articulating, testing, providing rules or conjectures for an infinite class of numbers (Canavarro, 2007; Kieran et al., 2016). The evidence that children in the early years can solve tasks thinking algebraically has led to changes in international school curricula, expanding the concept of Algebra in countries such as the United States and Portugal (Molina, 2011). In turn, Kieran et al. (2016) also point out that it is not enough for teachers to understand well mathematics only, they need to acquire experience in looking into the students' thinking, they must (re)orient their practices and beliefs to develop and listen to the mathematical ideas that students bring.

In Brazil, Trivilin and Ribeiro (2015) emphasize the need for offering continuing education to teachers who teach mathematics in the early years, since their professional knowledge has gaps regarding AT development. The results pointed out by Trivilin and Ribeiro (2015) show it is important to understand the different meanings of the equals sign, expanding the operational meaning beyond what is normally approached in schools. Such indications agree with the BNCC (Brasil, 2017) recommendations, thus giving even more topicality to the results we analyze in this research.

Given the understanding of the different meanings of the equality sign as one of the themes to be developed in the AT field, we consider from Ponte, Branco and Mattos's (2009) study the weight of the concept of equality in mathematics, since it plays a consistent role for us to grasp the concept of equivalence. These authors emphasize that "mathematical equality or equivalence is always in relation to a given property only" (p. 19). It is important to remember that, in mathematics, the equality relation is an equivalence relation with three properties: the symmetrical ($6+2 = 8$ or $8 = 5+3$ or $6+2 = 5+3$), the reflexive ($3 = 3$) and the transitive property ($2+3+4 = 4+3+2 = 5+4 = 9$) (Ponte, Branco & Matos, 2009).

In Kieran's (1981) work, three different meanings are pointed out for the equality sign: first, the operational meaning; second, the equivalence meaning; and third, the

relational meaning. The operational, the most focused meaning in the early years and, quite often, the only one, gives the student the idea that, after the symbol “=”, the result of an operation should always be placed and, generally, only a single quantity is accepted as true (for example, $4+13 = 17$). Under these circumstances, students are limited to the notion that the sign of equality is “a sign of doing something” (Behr, Erlwanger, & Nichols, 1980); an indicated action that means: it gives something or does something (Stacey & Macgregor, 1997); “an operator that transforms, for example, $3 + 4$ into 7 ” (Trivilin & Ribeiro, 2015). The second meaning of the equality sign, that of equivalence, allows us to establish many ways of representing 12 through numerical equalities, such as $12 = 6+6$; $8+4 = 12$; $12 = 10+2$, and indicates possibilities to work expressions like $7+3 = 2+8$, indicating a balance, an equivalence relationship between the terms “before” and “after” the sign.

It is essential to working on this meaning in the early years to enable students to understand the algebraic concepts presented in the subsequent grades, such as the concept of equation, which is largely discussed in the final years of elementary school (Ribeiro & Cury, 2015). Finally, the last meaning of the equals sign is the relational one, by which relationships between expressions are established, as well as the understanding and use of the properties of operations (addition and multiplication). In this case, the equality sign is fundamental for us to understand, for example, the expression $10+12+15 = 10+10+17$.

PROFESSIONAL LEARNING TASKS

When discussing the teacher’s professional learning and the Professional Learning Tasks (PLT), we start from Ball and Cohen’s (1999) study. The authors suggest that professional learning be seen from a curriculum structure in which it is considered in the context of the development of a specific pedagogy for teacher education. In this case, the education focuses on new skills to study practice based on questions centered on practice (such as the use of records, tasks and activities, class videos, planning, events, etc.). Such an approach seems to make it possible to open comparative perspectives on the practice (opportunity to learn from the practice of the other) and, finally, to contribute by putting personal and collective questions (opportunity to (re)signify/transform beliefs and practices). Ball and Cohen (1999) also present their pedagogical model of professional learning for teacher education, which is structured in three pillars and their key components: the PLTs, the nature of the discussions fostered by the unfolding of the PLTs, and the significant role of the trainers who would facilitate such tasks and discussions.

In this context, we seek to identify how teachers react when they experience (good) opportunities to learn, based on the following assumptions: (i) the topic they teach (meanings and connections with everyday life and not just procedures and information); (ii) students’ knowledge (how they think, how they learn, why they make mistakes, how

to listen carefully and how to help them move forward); (iii) the need to develop the ability to overcome social and ethnic differences with sensitivity to proceed with the necessary adjustment and adaptation to reach each student and seek strategies for all of them to learn (Ball & Cohen, 1999).

On the other hand, the studies by Silver et al. (2007) bring an analysis that illustrates some ways in which teachers can learn mathematics collectively. The authors propose a project with a format of professional learning opportunities, which consists of a four-step sequence, namely: 1) *Opening Activity Problem Solving*, an opening activity in which teachers are invited to solve a non-trivial mathematical problem; 2) *Individual Reading and Analysis of the Case*, when individual reading and analysis of a class narrative is proposed; 3) *Collaborative Case Analysis and Discussion*, in which there is a collective discussion of the case narrated, when the teachers raise the questions of the case and correlate them with their own teaching; 4) *Collaborative Lesson Planning and Debriefing*, culminating in the collaborative planning of a class, its subsequent development and analysis together. Those discussions allow us to say that the possibilities of professional learning are established in collective training contexts.

Finally, it is important that the PLTs consider “authentic samples of practice” (Smith, 2001) that is, the use of materials extracted from actual classes, in the classroom, such as videos, audios, observation, students’ protocols, among others, that can provide opportunities for professional learning by opening space for criticism, questioning and investigations.

RESEARCH CONTEXT AND METHODOLOGICAL PROCEDURES

This study was developed in the context of a continuing teacher education process, in a public municipal school in São Paulo. The meetings were structured in 14 face-to-face work sessions from August to October 2018, conducted by the first author, as a researcher and teacher educator (TE), and six early childhood education teachers. The data were collected in this context.

In the first meeting of the formative process, a questionnaire conducted with the teachers provided us with opportunities to survey and identify their previous mathematical knowledge about the different meanings of the equals sign. The other objective was to discuss theoretical and methodological elements for the foundation and deepening of the teachers’ knowledge. Meanwhile, in the subsequent meetings, we worked with the PLTs to mobilize and build mathematical knowledge to teach the different meanings of the equals sign in the early school grades. Considering the focus of this article, Table 1 shows a summary of meetings 6, 7 and 8, moments in which the PLTs analyzed and discussed in this article were developed.

Table 1*Overview of part of the research meetings*

MEETINGS	OBJECTIVES
6th meeting 09/03/2018	To develop PLT 1, aiming to identify whether teachers recognize the meanings of equivalence and the relational meaning of the equality sign. To survey the teachers' knowledge about students and their resolution strategies and possible difficulties/mistakes.
7th meeting 09/05/2018	To develop PLT 1 to enable the analysis of the students' fictitious resolution and different strategies to solve the task proposed, showing mathematical knowledge of the different meanings of the equality sign.
8th meeting 09/10/2018	To develop PLT 2, aiming to discuss students' anticipations and difficulties when performing a task, and the challenges of looking at the students' protocols and explaining their procedures and possible interventions.

In the meetings where the teachers worked with the PLTs, they were divided into two groups³, in a dynamic composed of two moments: (i) in the first moment, each group read, reflected, discussed, recorded their conjectures and the resolution of non-trivial mathematical tasks and, concomitantly, the TE, who circulated between the two groups, intervened occasionally; (ii) in the second moment, the plenary was opened, so that each PLT question/item and its resolutions and discussions in the smaller groups were shared with all participants.

From a methodological point of view, this research is part of a qualitative-interpretative approach (D'Ambrósio, 2004; Esteban, 2010), with data collected through three instruments and procedures: (i) a questionnaire, aiming to identify teachers' previous knowledge about the theme; (ii) the PLTs, with different approaches, to enable a variety of discussions and approaches related to specific knowledge, to students' knowledge and teaching processes and curriculum knowledge; and (iii) audio and video recordings, which were transcribed later, taken during the meetings.

The PLTs, the main instrument for data collection, had in their structure the purpose of raising and enabling the understanding of: (I) which mathematics knowledge the teachers mobilize to teach the different meanings of the equals sign; (II) which teaching practices could provide opportunities for their students' interaction and knowledge construction; and (III) which types of mathematical tasks and teaching approaches the teachers believed could enhance the teaching of the equals sign.

To construct the units of analysis, we took: (i) the initial questionnaire; (ii) the data produced through the transcripts of meetings 6, 7 and 8; and (iii) the written productions of the two groups of teachers when developing PLT 1 and PLT 2. This information was grouped, and an inventory organized by meeting, participant/group and instrument/

³ G1 group was composed of teachers Celeste (C), Luciana (L) and Kátia (K), and G2 group was composed of teachers Adionisia (A), Márcia (M) and Valdete (V). At the teachers' request, their real names were used.

collection procedure was created. When grouped, the information obtained from different sources would allow us to compare and analyze them articulately.

Based on the constitution of the inventory and a first round of analysis, two occurrences for an on-time and in-depth analysis were organized: one episode focused on the mobilization of specialized knowledge to teach the different meanings of the equals sign, and the other focused on teachers' knowledge of students and teaching. In the first event, there will be a total of 14 excerpts, of which 6 dialogues and 8 figures; in the second, there will be 9 excerpts, of which 4 dialogues and 5 figures will be discussed.

As a way of coding the information used in the analyses, the following procedure was adopted: The teachers' real names come first, followed by the description of the information, and the initial letter of the instrument (Q, for the questionnaire; T1 or T2 for PLT 1 or PLT 2, respectively; D, for transcriptions in small groups; and P, for transcriptions of plenary sessions), and, finally, the number and date of the meeting in which the data was obtained). For example, to identify information provided by Adionísia, at the 6th meeting, on 08/08/2018, during group discussions, we used: (Adionísia, D6, 08/08/2018).

FROM OPERATOR TO EQUIVALENT: EXPANDING THE TEACHERS' KNOWLEDGE ABOUT THE MEANINGS OF THE EQUALS SIGN

This first event presents and discusses the analyses of the teachers' mathematical knowledge to teach the different meanings of the equals sign to students in the early school grades.

From questions (11) "*In your opinion, who is right, Carla or Joana? How would you mathematically justify your choice?*" and (12) "*What do the students' responses reveal about the equals sign?*", a part of the Initial Questionnaire (Figure 1), we can apprehend that, taking the teachers' analyses of a mathematical task for the students, until then, they only recognized the operational meaning of the equals sign.

Figure 1

Excerpt Translated from the Questionnaire

Teacher Ana used the following task in her class presented above

TASK

1 - Check the alternatives and justify whether they are correct or not:
A) $2+2+3 = 5+2 = 7$

Read the conversations in pairs:
"Carla explained that alternative A was incorrect, because they had not added all the addends to result 21, reading " $3+2+2+5+2+7 = 21$ ". Joana said that the alternative was correct, and the answer was seven".

We can observe that for question (11), teacher Celeste (C) related the equals sign to the associative property of the addition, based on student Joana's answer. Based on student Carla's answer (question 12), teacher Celeste stated that, for this student, the equals sign had no meaning - even though she associated the equals sign with its meaning as operator. On the other hand, it is interesting to note that teacher (C) recognized the operational meaning of the equality sign in Joana's answer, although this was not the meaning the student had attributed it. However, when reflecting on student Carla's response, Celeste did not realize, precisely, that, although incorrectly, student Carla pointed out the operational meaning (Figure 2⁴):

Figure 2

Questionnaire Protocol (Survey data - Celeste, Q2, 08/08/2018)

11 Joana acertou, pois usou a propriedade associativa da adição, ou seja, não importa a forma como as parcelas foram associadas, a soma ou total será a mesma.

12 Para Carla o sinal de igualdade nada significa, já para Joana, ele tem valor, pois representa o resultado da adição das parcelas.

On the other hand, when analyzing the same questions, (11) and (12) (Figure 3), teacher Luciana (L) seemed to understand Carla's answer, including attributing understanding to what the student did. This can be seen in the way Luciana justified Carla's mistake. However, for Joana's response, Luciana stated that the student "understands the sign as the completion of the calculation", thus demonstrating a yet operational view of the equality sign (Ponte, Branco, & Matos, 2009; Trivilin & Ribeiro, 2015).

Figure 3⁵

Protocol for Questions 11 and 12 of the Questionnaire (Research Data - Luciana, Q2, 08/08/2018)

Enquanto resposta / resultado final Joana está correta. Porém a elaboração do raciocínio de Carla aponta para uma hipótese referente à soma de forma linear.

Para Joana o sinal de igualdade é compreendido como a finalização do cálculo, para Carla não há esse conceito, o que ocasionar a não da soma como "unite todas as parcelas".

⁴ 11) Joana got it right, because she used the associative property of addition, i.e., regardless of the way the addends were associated, the sum, or total, will be the same.

¹² For Carla, the equals sign means nothing; on the other hand, for Joana, it is valuable, because it represents the result of the summation of the addends.

⁵ Concerning her answer/result, Joana is correct. However, the elaboration of Carla's reasoning indicates a referring hypothesis - the linear summation.

For Joana, the equals sign is understood as the end of the calculations. Carla does not have the same conception, which caused their view of the sum as "unite all the addends."

We can also observe that, at the beginning of the work in groups, raised by PLT 1 (Figure 4), the meaning of operator of the equality sign remained resistant, which is reinforced in the discussions between the two teachers in G1. However, throughout the development of PLT 1, especially with the interventions of the TE, this changed and new meanings of the equals sign began to emerge.

Figure 4

PLT 1: The Mathematical Task

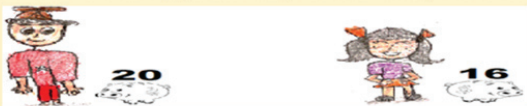
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PLT 1 – PART 1 – THE EQUALS SIGN

“Teacher Jane was analysing the answers of her 4th-grade students to the task proposed.

Know the task developed by her:

Auntie gave Artur and Cecilia the same amount of money. Artur decided to save 20 reais in his pig bank and keep some pocket money to take to school. Cecilia saved 16 reais in her pig bank and put some money aside to buy stickers.



Since the siblings received the same amount of money each, we can establish the equality:

20 + = 16 +

Determine the amount each child put aside to spend.
Explain how you got to the answer.

Note: Translated from the Original, in Portuguese

Although the teachers found correct mathematical answers to what was asked in the PLT, it is interesting to mention that they solved the task (Figure 4) by operating separately “what comes before” and “after” the equals sign. When observing this situation, to get the teachers to reflect on what they were doing, TE raised some questions:

TE - *Girls, I saw that you have answered everything and put several possibilities for resolution, so, can I ask you something? You put 10 and 14; 4 and 8; 2 and 6... what is the relationship between each pair of numbers?*

C – *Because we only worked with the even numbers?*

TE - *But can you only put an even number? What if I put 15?*

C – *You could do it, too. [...]*

K – *20+15 equals 35.*

C - *Then it would have to be on the other side to give 35 as well. 16+__*

K – *19.*

TE - *And why did you calculate the total for each side? Couldn't it be that there is no relationship between what I put on one side and the other? [...]* And what relationship is there between what I put on this side [pointing to the quantity to

be added with the 20] *and what I put on this side* [pointing to the quantity to be added with the 16]? *What relationship can we establish?*

(Celeste, Kátia, Luciana, Teacher Educator, D6, 09/03/2018).

Despite the questions asked by the TE, the teachers continued calculating the result separately, on each “side” of equality, thus keeping the operational meaning of the equality sign (Ponte, Branco, & Matos, 2009). Following, the debate continued:

K - *But aren't you working with the sum?*

TE - *No, I want something beyond the sum. Look, you did this all the time* [pointing to $20+10 = 16+14 = 30$] *you wanted to balance and calculate. Everything you calculated on one side of the signal, you tried to find the balance by calculating on the other.*

C - *Yes, we added to find the balance on the other side.*

TE - *Yes, but is there a way to determine the amount to be placed on the other side, without having to add each side separately? Would there be any regularity that we could see, between the number to be placed on one side and its relationship with what will be placed on the other? Look here, you put 10* [pointing to the first part of the equality $20 + \underline{\quad}$] *and here* [pointing to the quantity to be added to 16]?

K - *14.*

TE - *Here, did you put 4* [pointing to the first part of the equality $20 + \underline{\quad}$] *and here* [pointing to the quantity to be added to 16]?

C - *8.*

TE - *And then?*

C - *Ah, you're always adding 4!*

TE - *And why?*

K - *Wow, I hadn't realized that.*

C - *I've just realized it. Good question, and why?*

L - *Ah, it is because between these two there is this difference of 4* [pointing to 20 and 16].

TE - *And if there are 4 less... on the other side...*

L - *On the other, there will be 4 more.*

C - *So, if I put an odd one here* [addressing the TE], *for example $20+5$, which is 25. It would be $16+9$* [answering quickly].

TE - *Exactly!*

(Celeste, Kátia, Luciana, Teacher Educator, D6, 09/03/2018).

The teachers, then, started to perceive the regularity for any quantity to be added to the values on both “sides” of the equality sign (which, in the initial case of PLT 1, was 4 added to the value after the equals sign). With that, they broadened their look into the

equality sign, moving from the single-eyed operational meaning (“ $20+15$ equals 35 . Then it would have to be on the other side to equal 35 too”) to realizing its equivalent meaning (“So, we see here that $20+7 = 16+11$. Now it was quick. Yes, there is always a difference of 4 ”) (Kieran, 1981). We can say that there was construction in teachers’ mathematical knowledge (Ball, Thames & Phelps, 2008) and expansion of the meanings of the equality sign (Kieran, 1981; Ponte, Branco, & Matos, 2009). Yet, it seems possible to conjecture that teachers were in an increasing (re)structuring of their Specialized Content Knowledge (Ball, Thames, & Phelps, 2008).

After understanding the regularity that exists in the relationship of the quantity they placed on both members of equality and discovering a pattern in the relationship between the answers they gave, which are important AT components in the early school years (Britt & Irwin, 2011; Ponte & Branco, 2013), a new and relevant question arises. The teachers started to wonder whether they could generalize the pattern found to any other tasks: “*And will it always be 4? In any proposal?*” The search for a generalization can be seen below:

C – *And will it always be 4? In any proposal?*

TE – *In this case [Professor Jane’s task], yes. But what if we changed the number? And if the boy had kept 15 and his sister 10, would it be the same?*

[...] C – *15 and 10... Then, it would be 5. Is that it?*

TE – *Exactly.*

K – *Ah. It is from here!*

TE – *Yes, it is the relationship that is established, since the two received the same amount of values; so, if there are 5 more here, and 5 less on the other, to keep equivalence I have to consider this.*

K – *Wow, look, if I put $15+5$, I just put $10+10$, the difference is also 5.*

TE – *You stop looking only at the sum and start to establish a relationship of what you have on one side, before the equal sign, and on the other; after the sign. And you start to work on the central idea of seeing the equality sign with another meaning, not just the operational one, but as an equivalent and even relational.*

K – *Wow, very interesting. Now we look here and know the result there. [...] It shakes up the child’s mind a lot... it has shaken up mine.*

(Celeste, Kátia, Luciana, Researcher/Teacher Educator, D6, 09/03/2018).

Also discussing the teachers’ mobilization and expansion of meanings of the equals sign, the data that emerged during the work with another part of PLT 1 was analyzed (Figure 5). This part presented real and fictitious responses from teacher Jane’s students, when performing a mathematical task proposed (Figure 4). To analyze the answers, the teachers explored the mathematics involved in them; and they went on, mobilizing and expanding also a type of Specialized Content Knowledge (Ball, Thames, & Phelps, 2008), speaking of other meanings of the equals sign (Kieran, 1981; Ponte, Branco, & Matos,

2009) and recognizing the AT in the early school years (Britt & Irwin, 2011; Ponte & Branco, 2013).

Figure 5

PLT 1: Real and Fictitious Students' Resolutions

<p>Carlos, Joaquim and Cristina</p> <p>“Artur took 10 reais to school and Cecilia put aside 14 reais for her stickers. We thought this way, if they received the same quantity, so Cecilia spent 4 reais more than her brother, and we found that he took 10 reais because at school you can take up to 10 reais. So, we have $20 + 10 = 30$ and $16 + 14 = 30$.”</p>
<p>Thiago, Fernanda and Carolina</p> <p>“Artur took 36 reais to school and Cecilia spent the same 36 reais in stickers, because they had the same quantity. We arrived at this answer by putting together the values that appear in the account $20 + 16$.”</p>
<p>Thais, Matheus and Murilo</p> <p>“Artur took 5 reais to school and his sister spent 9 reais in stickers. We thought that, if both received the same amount, and he saved 4 reais more in the pig bank, so Cecilia had 5 reais plus 4 reais to spend on her stickers. We arrived at this answer by adding $20 + 5 = 16 + 9$, because Artur saved 4 reais more than his sister.”</p>

Fonte: Lillian C. S. Barboza

Note: Translated From The Original, In Portuguese

Teachers Adionísia (A), Marcia (M) and Valdete (V), from group 2 (G2), made their analyses and conjectures:

M – *But then, in this case here, they didn't notice...*

A – *The equality.*

M – *Yes, they did not realize the equivalence. He ignored the equal sign as an equivalence.*

M – *So let's go back here [they reread Carlos, Joaquim, and Cristina's answer].*

A – *[...] They understood it. Not only did they perceive equality, but they did equivalence [...]. And he also realized that Cecilia has a difference of 4 reais.*

M – *Different from this, then?*

A – *Very different because he perceives equality. And this one goes there and adds up.*

M – *Look, it's right. They understand the reasoning and also discover the difference of 4 reais.*

(Adionísia and Márcia, D7, 9/5/2018).

From the discussions of teachers (A), (M) and (V), it became clear that they were using the equivalent meaning of the equality sign, which can be seen from their analyses of their students' responses. Although we had not previously presented the results of the teachers of group G2 (their answers in the initial questionnaire), they also did not, at that moment, show (explicit) signs of knowing and recognizing other meanings of the equality sign beyond the operational meaning.

Considering the entire course of discussions held by teachers (Ponte & Quaresma, 2016), fostered by both the way PLT 1 was conceived and developed (Ball & Cohen, 1999;

Silver et al., 2007) and the performance (Smith, 2001), it is possible to affirm that the teachers started to mobilize and (re)signify their mathematical knowledge (Ball, Thames, & Phelps, 2008) for the teaching of the different meanings of the equality sign (Kieran, 1981; Ponte, Branco, & Matos, 2009) – especially by expanding their understanding of the meanings of the equality sign from operator to equivalent.

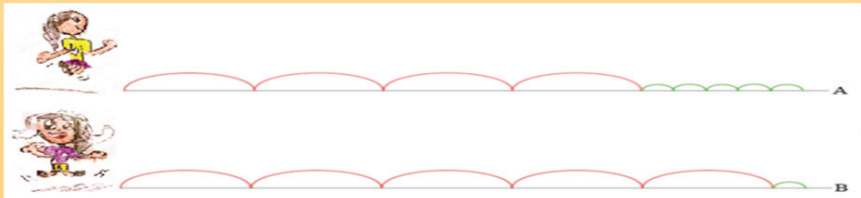
We discuss below some data collected from the work the teachers developed with PLT 2 (Figure 6).

Figure 6

PLT 2: *The Mathematical Task*

During PE classes, teacher Thais suggested that her class took two different paths, both with the same length. Each path had a number of leaps, all with the same length and indicated by the pink color, and a number of steps, also with the same length and indicated by the green color.

Vitória followed path A and Alice followed path B. Observe



How many steps correspond to the whole path? Explain how you got the result.

Adaptada de Ponte, Branco e Matos, 2009, p. 38.

Note: Translated from the Original, in Portuguese

The resolution and explanations G1 teachers, (C), (K) and (L) (Figure 8) presented, elaborated from the analysis of the students' fictitious and real answers (Figure 7), demonstrate the appropriation and the use of the equivalence meaning to develop what was requested in PLT 2. As we can see in Figure 7.

Figure 7⁶

PLT 2: *One of the Students' Resolutions*

Elisa e Fernanda

$$4 \text{ leaps} + 5 \text{ steps} = 5 \text{ leaps} + 1 \text{ step}$$

$$V = 4 \text{ leaps} + 5 \text{ steps}$$

$$A = 5 \text{ leaps} + 1 \text{ step}$$

$$5 - 1 = 4$$

$$V = 4 + 4 + 4 + 4 + 5$$

$$A = 4 + 4 + 4 + 4 + 4 + 1$$

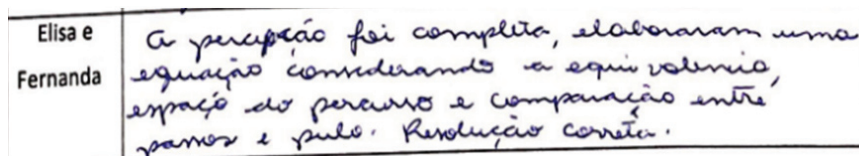
O percurso da professora tem 42 passos. A gente viu que Vitória tem 4 saltos e 5 passos e a Alice tem 1 salto a mais e 4 passos a menos, então descobrimos que o salto vale 4 passos e depois foi só contar +4 para cada salto.

⁶ Elisa and Fernanda - The teacher's path has 42 steps. We saw that Vitória has 4 leaps and 5 steps, and Alice has one leap more and 4 steps less, so we discovered that the leap is worth 4 steps, and so, we just counted +4 for each leap.

After analyzing the students' real and fictitious responses and resolutions, the teachers should answer, according to their professional experience, whether they considered them to be (in)correct. Thus, the response protocol generated by (C), (K) and (L), teachers in the G1 group, was:

Figure 8⁷

Protocol presented in the analysis of Figure 7 (Research data - Celeste, Kátia and Luciana, PLT 2, 09/10/2018)



Through their answers, G1 teachers reveal that they had appropriated the meaning of equivalence of the equality sign. Beyond that, the teachers (C), (K) and (L) delved into the students' responses (as shown in Figures 7 and 8) and pointed out more detailed mathematical justifications (for example, referring to the term "equation") and, unlike what they had done right at the beginning of the training process, for example, when answering the questionnaire, they not only evaluated the resolution as correct or incorrect. The teachers were expanding their professional learning (Ball & Cohen, 1999).

Besides the work in small groups, during the plenary there was also evidence that teachers (A) and (M) started to perceive the sign of equality with the meaning of equivalence:

TE – *So, that's it, do I really need to add the two sides together? What does this sign of equality mean? [...] In this task, which of the meanings of the sign of equality is most evident? Would that be the operational one?*

M and A – *No!*

TE – *Which one?*

M and A – *The one of equivalence.*

A – *This is not operational at all. It is actually equivalence, from one side to the other.*

(Adionísia and Márcia, P6, 09/03/2018).

Another evidence can be observed following the discussion during the plenary, as teachers (A), (M) and (V) paid attention to the fact that students find it difficult to

⁷ Elisa and Fernanda - The perception was complete, they elaborated an equation considering equivalence, space of the path and comparison between the steps and the leaps. Correct resolution.

recognize the equals sign beyond the meaning of operator, because this is the most used meaning in school (Ponte, Branco, & Matos, 2009; Trivilin & Ribeiro, 2015).

M – *We said that they ignore the equal sign as equivalence.*

TE – *Is that true? [...]*

A – *Yes!*

TE – *Because, what are they used to seeing?*

M – *One account only and the answer after the equal sign. It's not in the middle. After it, the result comes.*

V – *That is true.*

(Adionísia, Márcia and Valdete, P7, 9/5/2018).

We can conclude, albeit partially, in view of the evidence and analysis presented so far, that there was, from the part of the collaborating teachers in this research, mobilization and expansion of the meanings of the equality sign, especially in the sense of failing to recognize it only for its operational meaning and starting to understand and use its meaning of equivalence, too. The teachers also recognized the students' possible mistakes, dimensioning their nature, characteristic of the SCK (Ball, Thames, & Phelps, 2008).

We observe that the teachers could (re)signify their mathematical knowledge for teaching (Ball, Thames & Phelps, 2008), from the analyses and discussions around real and fictitious situations arising from teaching practice (Ball & Cohen, 1999; Smith, 2001; Silver et al., 2007), and from the PF's specific interventions. They also showed that the mathematical tasks commonly provided at school can be a means of reinforcing only the operational meaning of the equality sign (Ponte, Branco, & Matos, 2009; Trivilin & Ribeiro, 2015).

ANTICIPATING STUDENTS' RESPONSES AND THEIR RESOLUTION POSSIBILITIES: FROM DISCUSSIONS TO THE TEACHING PRACTICE

Now, with this second episode, the focus of analysis and discussion started including potential difficulties that students of the early school years may present with mathematical tasks - such as those explored in the previous section -, as well as possible interventions that they, the teachers, can carry out aiming to help their students overcome difficulties, such as those they can normally present.

Resuming the development of PLT 1, when asked to answer the question "*What difficulties can students in the 4th and 5th grades present when performing this task?*" (referring here to the mathematical task shown in Figure 4), we call attention to what the teachers (A), (M) and (V) brought of reflections, when starting their discussion:

M – *Difficult, isn't it?*

A – *But it is not known how much she got?*

A – *Ah, so, we are the ones who can establish it.*

V – *But the issue is the child's difficulty.*

A – *So that's it, we are going to establish the value, so another child can establish something else. It will not have a single result.*

M – *So there will be no single result! So, "what difficulties can children present" is to find that out. Because the child thinks that they will have a unique answer for everyone, and they won't have it. I didn't realize it, why would they?*

(Adionísia, Márcia and Valdete, D6, 09/03/2018).

At first, the teachers could not anticipate difficulties explicitly related to the different meanings of the equality sign, but, on the other hand, they affirm that it was a difficult task, mainly because they did not have only one correct answer (a single answer).

These teachers were also invited to reflect on whether or not to use the task in question (Figure 4) in a math class and, if so, they needed to decide which school grade would be most appropriate and how it should be developed with students.

They started by arguing that there should be a "step by step" of the teacher's actions, that should start with the formation of work pairs and that the teacher should propose some questions for the students' engagement in the task. Teachers were mobilizing two types of professional knowledge for teaching, namely Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) (Ball, Thames, & Phelps, 2008). The G2 teachers pointed out the importance of asking students: "*Why do you think this sign of equality is in the middle of this sentence?*", a reflection arising from Adionísia's intervention (D6, 09/03/2018). However, the teachers highlighted, as noted in the discussion below, the centrality of the teacher's role, emphasizing that a collective reading of the task with the students was essential:

M – *Yes. Leaving them to read on their own may not help.*

A – *And what do you think of the question: What are the possibilities of making sentences the same?*

V – *I think this is fundamental. You are going to do it in your classroom, right?* [Referring to another moment in the discussion, in which teacher (A) states that she would work on the mathematical task (Figure 4), on her own initiative, in her classroom]. *Then, you will tell us.*

A – *But after they realize that there are many possibilities, they need to realize the regularity, because that is something else.*

M – *Then, from there, they can realize that they are equal and discover the other values. It is the kind of problem that needs good direction, questions, conversation. Working in pairs is essential for exchanging with each other. And yet the teacher must pass by, interfere, ask questions.*

(Adionísia, Márcia, Valdete, D6, 09/03/2018).

Although the meaning of equivalence of the equality sign does not seem to be the focus of the teachers' discussions, they now begin to consider it as part of the task, presenting its relation with students' possible difficulties and with teaching strategies that could be mobilized to seek to overcome such barriers.

When we started to analyze the work done by G1 group regarding the students' difficulties concerning the mathematical task presented in Figure 6, teachers (C), (K) and (L) recorded their answers and presented them, as seen in Figures 9 and 10:

Figure 9⁸

Protocol Presented at TPLT 2 (Research data - Celeste, Kátia, and Luciana, PLT 2, 09/10/2018).

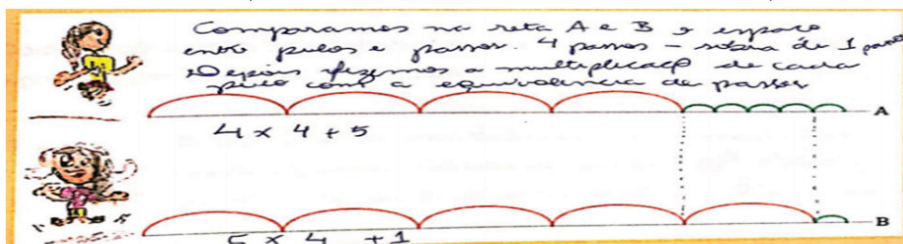


Figure 10⁹

Protocol Presented - Possible Difficulties for PLT 2 Students (Research data - Celeste, Kátia and Luciana, TAP 2, 09/10/2018).

Cu principio os alunos poderão ter dificuldade em perceber a diferença entre passos e pulos. Também a "falta" de equivalência comparando quantos passos "caem" num pulo.

The teachers highlighted some difficulties that students could face, such as relating "leaps" and "steps", but there is a deepening of their reflections during the discussions they develop in the work in small groups (Ball & Cohen, 1999). This can be seen in the following excerpts:

C – Look, based on that, we can already elaborate some questions to ask, "is the space occupied by the leap the same as a hop?" "Does the same number of leaps and hops fit in a space?"

⁸ We compared in line A and B the space of the leaps and steps. 4 steps – of 1. After that we multiplied each leap with the equivalent in steps.

⁹ At first, the students may find it difficult to realize the difference between steps and leaps. There will also be the "lack" of equivalence, comparing how many steps "fit" into a leap.

L – *That's right.*

C – *Both must cover the same distance, cover the same path. Will Alice's number of jumps be the same as Vitória's path?*

L – *Hum.*

C – *Because there are only leaps and one step here. And here there are leaps and more steps. Won't they get mixed up with the quantity of steps here?* [pointing to path A in Figure 9].

K – *I believe so.*

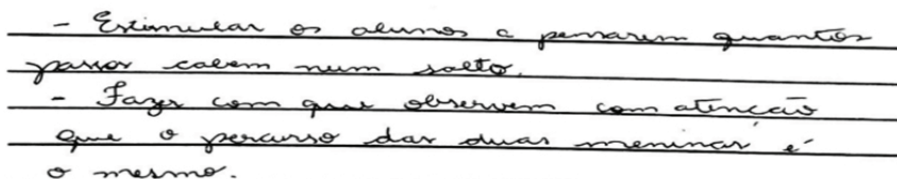
C – *Because they can look at a lot of steps here and say "Look, teacher, there are few leaps here, and are a lot of steps there, so this one walked more."*

(Celeste, Kátia and Luciana, D8, 09/10/2018)

Teachers, (C), (K) and (L) suggested how to anticipate possible difficulties related to the task (which is observed in the previous transcript and in those that follow Figure 11). Besides anticipating students' potential difficulties, teachers (C), (K) and (L) started to think about what actions they could take to explore equivalence, for example, based on the question they would ask students: *"think how many steps fit in a leap."* And they intended that students observed that *"the path of the two girls is the same,"* that is, the paths are equivalent.

Figure 11¹⁰

Protocol Presented- Actions to Explore the PLT 2 Equivalence (Research data - Celeste, Kátia and Luciana, PLT 2, 09/10/2018).



- Estimular os alunos a pensarem quantos passos cabem num salto.
- Fazer com que observem com atenção que o percurso das duas meninas é o mesmo.

In view of the discussions of teachers (C), (K) and (L) and teacher (C)'s last statement *"Because they can look at a lot of steps here and say: Look, teacher, there are few leaps here and there are lots of steps there, so this one walked more,"* the TE came closer and asked:

TE – *Could this be an error right away?*

C – *Yes, I think so.*

¹⁰ - Encourage the students to think how many steps fit in a leap.

- Make them observe carefully that the path of the two girls is the same.

TE – *And then, what would they disregard? What information about the problem would they disregard?*

C – *That is the same amount.*

L – *The equality of the path. It's the equivalence here.* [Pointing to the task]

TE – *And what changes?*

L – *That one ended with more steps only.*

C – *And the other with more leaps. That's all.*

TE – *And the length was the same. And we are drawing attention...*

C – *Of equality.*

TE – *Yes, of the equivalence that exists and that can be written in different ways, giving the same result.*

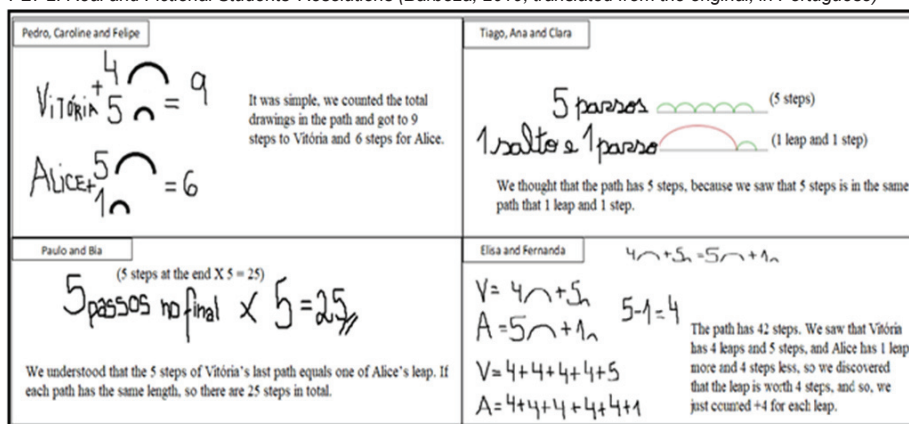
(Celeste, Kátia and Luciana, D8, 09/10/2018).

From the teachers' work with this part of PLT 2, besides a collective involvement in the search for a solution to an impasse identified by them as potentially emerging in the classroom (Smith, 2001), we can identify a mobilization, in an integrated and simultaneous way, of the different domains of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), especially with regard to the meaning of equivalence of the equality sign (SCK); students' difficulties with this content (KCS); and how they would intervene, to contribute to overcoming such difficulties (KCT).

Figure 12 presents students' answers to the mathematical task under discussion. Teachers were invited to analyze the answers, to allow them to now work with the students' answers (and no longer anticipating possible difficulties). During the analysis of the students' answers, teachers (C), (K) and (L) recognized in some of the resolutions the mistakes they themselves made when solving the task.

Figure 12

PLT 2: Real and Fictional Students' Resolutions (Barboza, 2019, translated from the original, in Portuguese)



The synthesis of the analyses and discussions developed by teachers (C), (K) and (L), of the G1 group, is recorded in Figure 13:

Figure 13¹¹

Protocol Presented – Analysis of Resolutions PLT 2 (Research data - Celeste, Kátia and Luciana, PLT 2, 09/10/2018).

Pedro, Caroline e Felipe	Esses desconsideraram a equivalência entre passos e saltos, realizando a pontuação sem comparação. A resolução está incorreta.
Tiago, Ana e Clara	Perceberam a equivalência entre passos e saltos, no entanto, não analisaram o percurso todo. A resolução está incompleta.
Paulo e Bia	Contaram os passos, perceberam o percurso porém não notaram a diferença de um passo ao final do percurso, o que os levou a igualar passos e pulos. Resolução incompleta.
Elisa e Fernanda	A percepção foi completa, elaboraram uma equação considerando a equivalência, espaço do percurso e comparação entre passos e pulos. Resolução correta.

Although, due to space restrictions, excerpts from the evidence of the G2 teachers' work are not presented here, we observed that the two groups considered the last resolution correct and all teachers understood that the girls reached 21 steps in the total and then added the paths. They also emphasized that students Elisa and Fernanda understood the equals sign with the meaning of equivalence and found the equivalence of the path.

With this, other partial results were synthesized, and will be put together with those presented in the previous section, for the conclusions and the final considerations in the next and last section. At this stage of the research, during the development of the PLTs 1 and 2, we observed that teachers, increasingly started to (i) emphasize the importance of anticipating difficulties that students may present when facing challenging tasks (such as those that have been proposed); (ii) propose interventions that they can carry out during the students' work phase; (iii) recognize that, possibly, the mistakes they had made could be the same as the students', and that the way in which the content in question is commonly treated at school can reinforce and maintain such mistakes.

Therefore, we emphasize that it is not enough for teachers to understand mathematics well "only", they must get used to looking at the students' thinking, they must (re)orient

¹¹ Pedro, Caroline and Felipe - They disregarded the equivalence between the steps and leaps, adding the terms without comparing. The resolution is incorrect.

Tiago, Ana and Clara - They perceived the equivalence between the steps and leaps; however, they did not analyse all the path. The resolution is incomplete.

Paulo and Bia - They counted the steps, realizing the path, but they did not notice a one-step different at the end of the path, which made them to equal steps and leap. Resolution incomplete.

Elisa and Fernanda - The perception was complete, they elaborated an equation considering the equivalence, space of the path and comparison between steps and leap. Correct resolution.

their practices and beliefs to develop and listen to the mathematical ideas students bring (Kieran et al., 2016).

CONCLUSIONS AND FINAL CONSIDERATIONS

To respond in this article to the objective of *identifying how professional learning tasks based on teaching practice contribute to the mobilization and expansion of the algebraic thinking of early childhood teacher, regarding the different meanings of the equality sign*, a qualitative-interpretative research was developed, with data collected in a process of continuing education offered to six early childhood teachers of the municipality of São Paulo. Based on the principles adopted, the proposal for continuing education was designed and developed in a space for collective exchanges and learning, a space that allowed teachers to study, share experiences, discuss and reflect on their practices, taking as a mathematical theme the different meanings of the equals sign and the development of algebraic thinking.

From the analyses developed in the previous sections, especially as a result of the work with the PLTs throughout the formative process, we concluded that the teachers started to understand the different meanings of the equals sign; to anticipate the students' responses; and to analyze the students' fictitious and real resolutions. In solving the mathematical tasks they had proposed to their students, the teachers recognized their own mathematical knowledge limitations on the subject, which, according to them, could hinder and influence students' mistakes and understanding in the task of recognizing and assigning other meanings to the equals sign, beyond the operational meaning.

We understood that the PLTs alone do not constitute possibilities to mobilize and expand the mathematical knowledge of teachers to teach mathematics in the early years. There is no naive view that such improvement and deepening of teachers' professional knowledge will be carried out only by proposing "good" PLTs. However, the data obtained in this study point to the potential of working with PLTs and the importance of encouraging the reflection of the teacher who works with them, which was complemented by questions the trainer asked, and with the discussions triggered among all participants in the formative process.

Based on the experiences of this research, we can also say that (new) contexts of continuing education for teachers who teach mathematics, such as the one presented here, must be considered, so that teachers can continue to learn throughout their teaching practices. Also, this research made it possible to work within the thematic unit/axis "Algebra", which is part of the BNCC (Brasil, 2017) and the new São Paulo *Curriculo da Cidade* (PMSP, 2017), discussing the different meanings of the equality sign.

It was not the purpose of this article to explore the teacher's practice and its relationship with the professional learning triggered from the educating process that was developed. This is explored in Barboza, Pazuch, and Ribeiro (2020) and may complement the possibilities and advances achieved so far. We conclude by pointing out gaps that

need more in-depth investigation in further studies, such as the role of the knowledge of teacher trainers of mathematics teachers or teachers who teach mathematics, and whether and to what extent discursive interactions contribute to teacher learning, among others. Such gaps must be explored beyond the field of Algebra and Algebraic Thinking.

CONTRIBUTION STATEMENT

LCSB prepared the material, developed, collected, and analyzed the data and AJR and VP followed the whole process and contributed with directions and analyses. All authors discussed the results and contributed to the final version of the manuscript.

DATA AVAILABILITY STATEMENT

The data of this research will be made available by the authors upon reasonable request.

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