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Knowledge for Teaching Justifications and Proofs or Future Mathematics Teachers

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ABSTRACT

This article presents part of the discussions and results of a doctoral research project that analyzed issues related to the initial education of future math teachers, concerning the selection, organization, and preparation of situations that promote students' academic learning of fundamental ideas related to justifications and proofs. This study involves a group of ten pre-service teachers who attend the Federal University of Sergipe. The first phase of data collection involved the application of diagnostic instruments. The second phase, named the 'training process', was carried out following principles of the Design Experiments methodology and aimed to investigate how activity sequences exploring knowledge about proofs, from a didactic and curricular perspective, foster new meanings for learning concepts and attitudes. To consider theoretically the knowledge that a mathematics teacher should possess, we considered the categories by Ball, Thames and Phelps. The answers by future teachers to the diagnostic instruments revealed a certain cautiousness regarding the inclusion of proofs in school curricula. Discussions and reflections on learning situations, proposed during this phase, broadened the knowledge base necessary for mathematics teachers and gave new meaning to working with proofs.

Keywords: Initial Education for Mathematics Teachers; Knowledge for Teaching; Justifications and Proofs.

Conhecimentos para o Ensino de Argumentações e Provas de Futuros Professores de Matemática

RESUMO

Este artigo apresenta parte das discussões e resultados de uma pesquisa de doutoramento cujo objetivo foi refletir sobre questões relacionadas à formação inicial de futuros professores de Matemática no que se refere à seleção, organização e elaboração de situações que favoreçam a

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aprendizagem de alunos da Educação Básica sobre ideias fundamentais relativas às argumentações e provas. Trata-se de um estudo que envolveu um grupo de dez estudantes do curso de Licenciatura em Matemática de um campus da Universidade Federal de Sergipe. A primeira fase da coleta de dados constituiu-se pela aplicação de instrumentos de caráter diagnóstico. A segunda fase, denominada de Formação, foi realizada segundo princípios da metodologia *Design Experiments* e teve a finalidade de investigar se sequências de atividades que explorem conhecimentos sobre provas sob o ponto de vista didático e curricular podem favorecer a ressignificação do processo de ensino de conceitos e atitudes concernentes a esse tema por futuros professores. Em relação à fundamentação teórica, no que diz respeito aos conhecimentos que devem ser de domínio do professor de Matemática, foram consideradas as categorias estabelecidas por Ball, Thames e Phelps. As respostas dos licenciandos aos instrumentos diagnósticos revelaram certa cautela a respeito da inclusão de provas nos currículos da Educação Básica. As discussões e reflexões sobre as sequências, propostas durantes essa fase, ampliaram a base de conhecimentos necessários ao professor de Matemática para exercer a docência e a ressignificação do trabalho com provas.

Palavras-chave: Formação Inicial de Professores de Matemática; Conhecimentos para o Ensino; Argumentações e Provas.

INTRODUCTION

This article presents part of a doctoral research project with the goal of analyzing the knowledge necessary for teachers with respect to content, didactics and the curriculum in order to teach notions and procedures regarding proofs in Basic Education.

It is worth noting that the meaning of proofs in this study is broad: it includes the first justifications, experimentations and empirical verifications up to the demonstrations, which are the rigorous or formal proofs.

We believe this research to be relevant, especially since including initial classwork with justifications and proofs in Primary Education is recommended by some Brazilian curricula, such as those by São Paulo (2010) and Pernambuco (2013), similarly to other countries, such as England and France.

> There are some trends that have been gaining momentum in some curricula: the inclusion of classwork with justifications and proofs starting in Primary Education. Though there is no absolute consensus in this respect, since conceptions on the meanings in this study are not exactly the same, these recommendations are reasonably explicit in the curricula of some countries. (Pietropaolo, 2005, p.98)[1]

The National Common Curricular Base, for example, recommends proofs of the Pythagorean Theorem in the 9th year of Primary Education, as shown by skill 13: To prove

[1] Original text: – Há uma tendência que vem ganhando força em alguns currículos: a inclusão de um trabalho com argumentações e provas já a partir do Ensino Fundamental. Apesar de não haver um absoluto consenso a esse respeito, pois as concepções sobre os significados desse trabalho não são exatamente as mesmas, tais recomendações estão razoavelmente explícitas nos currículos de alguns países.

metric relations of the right triangle, among them the Pythagorean Theorem, including the use of similar triangles (Brasil, 2017, p.317).

On the other hand, the implementation of new recommended curricula requires, according to researchers, changes in the initial and continuing training processes for Mathematics teachers. Pietropaolo (1999) claims, for example, that the processes of establishing curricular proposals for Mathematics, such as that of the São Paulo State Secretary of Education in 1988, were ineffective, because they found that many teachers lacked the knowledge in this area, in addition to having highly ingrained concepts, beliefs and values, inadequate teacher training programs, and books that do not incorporate new classwork options. All of this makes the process slow, with almost imperceptible advancements and distortions in the application of new ideas, frequently undermining the teaching and learning process.

It is also worth mentioning that the *Sociedade Brasileira de Educação Matemática* (SBEM, "Brazilian Society of Mathematics Education"), with respect to the teaching degree, in 2003, prepared a document that is still very current because it defends that, in practice, there are problems, previously identified by educators and researchers of Mathematics teacher training, that have yet to be overcome, such as the following.

> – The Mathematics teacher profile required today: should be a professional that is highly competent in formulating questions that encourage students to reflect, in addition to being creative when creating mathematically rich learning environments and situations.
>
> – The development of a unique identity for courses: course identities should be developed based on constitutive elements of the process of developing professional knowledge, such as: making connections between academic background and professional practice, emphasizing the didactic-pedagogical knowledge of the mathematics to be taught, and, during the teaching degree program, promoting investigative practices that promote the connection between theory and practice.
>
> – The selection of content and how approach it: teaching degree programs should assume a perspective that includes preparation for the teaching profession, which involves special treatment of mathematical content in basic education, in addition to content that broadens mathematical knowledge, such as: content from Differential and Integral Calculus, Mathematical Analysis, Algebra, Geometry, Statistics, Combinatorial Analysis, Probability, etc., which should be chosen so that they provide the teacher-in-training with broad, consistent knowledge linked to Mathematics (SBEM, 2003, p.4-10).[2]

[2] Original text: – O perfil de professor de Matemática exigido hoje: deve ser um profissional com grande competência para formular questões que estimulem a reflexão de seus alunos, além de ser criativo para criar ambientes e situações de aprendizagem matematicamente ricos.
– A construção da identidade própria dos cursos: a identidade dos cursos deve ser construída com base em elementos constitutivos do processo de construção do conhecimento profissional como: vinculação da formação acadêmica com a prática profissional,

We believe that these discussions presented by SBEM (2003) conceive of Mathematics Teaching Degree programs from the perspective of rupturing the dichotomy between pedagogical knowledge and specific knowledge, establishing the inextricable connection between theory and practice. In other words, teachers are trained in a perspective of Mathematics Education. This proposal by SBEM proposes that students and future teachers need to experiment, discuss and model situations similar to those that their future students will experience.

To develop our study, we chose the course of Special Topics in Teaching Mathematics as the space for our discussion. This course is part of the elective courses in the Mathematics Teaching Degree Program at the Federal University of Sergipe (UFS) – Professor Alberto Carvalho Campus. Moreover, like all courses labeled Topics, there is a specific syllabus and schedule proposed by the acting professor and approved by the collegiate body before the start of the semester. It is a 4-credit (60 hours) course, according to the schedule suggested, discussing topics related to our research topic, the use of proofs in Basic Education.

This study requested an ethical evaluation by the ethics committee (CEP/CONEP[3]), was approved under the number 982.198, and included the participation of 10 students from the Mathematics Teaching Degree program.

Our group included ten students enrolled in this course. The average age of these subjects was 22.5. Most came from public schools, are in the last semester of the Mathematics Teaching Degree program and are ready to work as Mathematics teachers.

In this study, students are referred to using the letters (A), (B), ... (J), in order to preserve their identity.

In order to contribute a proposal that promotes a reflection on including mathematical proofs in Basic Education, we proposed an investigation of the following question: To what extent can a sequence of activities that explores different types of proofs broaden the knowledge base for teaching this topic?

To answer this question, we initially proposed that the subjects of this study answer our questions in order to determine their knowledge regarding proofs and their conceptions on teaching them in Basic Education. We call this first phase of the research Diagnosis. Afterwards, we organized and developed two Learning Situations together with the students – one on the Pythagorean Theorem and another on Diophantine Equations, which formed the foundation of the *Design Experiment*, our chosen methodology. We used the

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ênfase no conhecimento didático-pedagógico da matemática a ser ensinada e promover, durante a licenciatura, práticas investigativas que promovam a articulação entre teoria e prática.

– A seleção de conteúdos e sua abordagem: os cursos de licenciatura devem adotar uma perspectiva que inclui a preparação para a docência, o que compreende o tratamento especial aos conteúdos matemáticos da educação básica, além de conteúdos ampliadores do conhecimento matemático, como: os conteúdos do Cálculo Diferencial e Integral, da Análise Matemática, da Álgebra, da Geometria, da Estatística, da Análise Combinatória, da Probabilidade etc., que devem ser selecionados de forma a possibilitar, ao professor em formação, conhecimento amplo, consistente e articulado da Matemática.

[3] Research project funded by CAPES under the number 99999.001706/2014-04.

term *Training* for this phase. Due to space limitations, we will present only part of the training in this article, which addresses the Pythagorean Theorem.

THEORETICAL FOUNDATION

Before we proceed to the results analysis of the diagnostic instrument and present the *Design* elaborated to promote a discussion on the process of addressing and developing proofs in Basic Education and a discussion of the data collected over the course of the Training, we must discuss some ideas that served as the foundation for this task.

Our analysis was developed by taking into consideration the domains by Ball, Thames and Phelps (2008) regarding the knowledge necessary for the teacher to teach, namely Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Horizon Content Knowledge (HCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), Knowledge of Content and Curriculum (KCC).

Common Content Knowledge refers to the mathematical knowledge that everyone learning a determined topic should have, and is not specific to those who work in the teaching profession. Regarding the Pythagorean Theorem, we expect individuals to apply this relation in order to solve problems in different contexts. This is necessary knowledge for teachers, but it is not exclusive to them.

Specialized Content Knowledge refers to specific knowledge for a teaching activity and, therefore, is not necessarily part of the repertory of a mathematics researcher, who produces new knowledge in mathematics, or that of the general public. In relation to the Pythagorean Theorem, we expect a mathematics teacher to know how to construct a sequence of empirical proofs (such as the pieces of squares built on the legs and the hypotenuse) which leads students to a formal proof.

All teachers should know how to justify the procedures that they teach, to have ways of providing meanings for the topics discussed and to have explanations and illustrations for the concepts that they address.

Another aspect that the Michigan group takes into consideration is Horizon Content Knowledge. This knowledge is necessary for the teacher to be able to highlight the key points of the content, propose problem situations that enable students to make connections and guide them in order to advance their own conjectures, preserving mathematical principles and familiarizing themselves with the language and the unique structure of the course.

We can identify elements of Horizon Content Knowledge in the sequence proposed for teaching the Pythagorean Theorem. One of the activities proposed for teaching this Theorem enables us to discuss incommensurable segments, for example. Making connections between elements is part of Horizon Content Knowledge, indispensable for teaching practices.

Without a doubt, the proof is a way to familiarize students with how mathematics validates its results, the language to communicate these results and the validation process.

Ball, Thames and Phelps (2008) highlight Knowledge of Content and Students, which claims that student understanding of Mathematics is related to experience. This enables the teacher to predict and interpret errors and search for strategies to overcome them.

Regarding Knowledge of Content and Teaching, Ball, Thames and Phelps (2008) refer to the fact that the teaching profession requires the teacher to select, organize and elaborate activities. This knowledge involves an analysis of the advantages and disadvantages of approaches and representations, as well as different methods and procedures.

And to complete the Knowledge necessary for the teacher to teach, Ball, Thames and Phelps (2008) present Knowledge of Content and Curriculum, which is necessary for teachers to help them make connections between the content taught and other content in the curricula for previous and following years, as well as content studied simultaneously in other courses. Included in this category is knowledge of curricular guidelines and curricular recommendations for introducing and developing content.

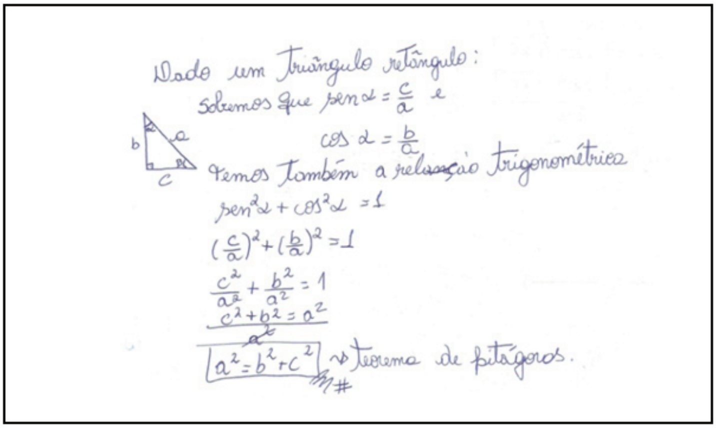
Regarding the Pythagorean Theorem, the teacher needs to emphasize not only solving problems that directly involve this relation, but also proposals that benefit the future study of topics, such as trigonometry and analytical geometry.

DATA COLLECTION AND ANALYSIS

Pre-service Teacher Knowledge in Relation to the Pythagorean Theorem

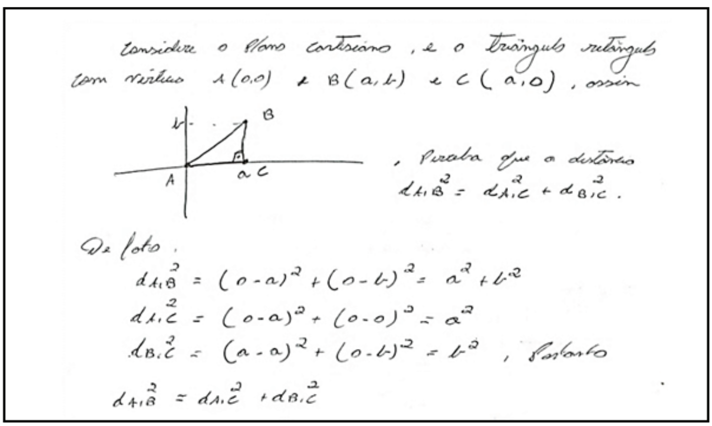
To analyze the teachers' knowledge on proofs in Basic Education, we proposed, among other things, that they talk about and present a proof for the Pythagorean theorem, in addition to indicating strategies for teaching this theorem.

In their attempts to prove the Pythagorean Theorem, the student-teachers presented empirical verifications and generalizations based on specific cases. However, the most common practice was mistakenly using the theory as an argument. This can be observed in the following protocols.



Dado um Triângulo retângulo:
Sabemos que $sen\,\alpha = \frac{c}{a}$ e
$cos\,\alpha = \frac{b}{a}$
Temos também a relação trigonométrica
$sen^2\alpha + cos^2\alpha = 1$
$(\frac{c}{a})^2 + (\frac{b}{a})^2 = 1$
$\frac{c^2}{a^2} + \frac{b^2}{a^2} = 1$
$\frac{c^2 + b^2 = a^2}{a^2}$
$\boxed{a^2 = b^2 + c^2}$ → teorema de pitágoras.

Figure 1. Pre-service Teacher (F) Protocol.



Considere o Plano cartesiano, e o triângulo retângulo com vértices A(0,0) e B(a,b) e C(a,0), assim

, Prova que a distância $d_{A,B}^2 = d_{A,C}^2 + d_{B,C}^2$.

De fato,
$d_{A,B}^2 = (0-a)^2 + (0-b)^2 = a^2 + b^2$
$d_{A,C}^2 = (0-a)^2 + (0-0)^2 = a^2$
$d_{B,C}^2 = (a-a)^2 + (0-b)^2 = b^2$, Portanto
$d_{A,B}^2 = d_{A,C}^2 + d_{B,C}^2$

Figure 2. Student (G) Protocol.

Pre-service Teacher (F) used trigonometric identity and student teacher (G) used the distance formula. Both used a variation or application of the Pythagorean Theorem in the body of the proof; in other words, they used the theory to "prove" the result. However, it is possible that the lack of knowledge on the part of the student teachers was due to not knowing that these formulas are a result of the Pythagorean Theorem, and not due to believing that they could use the theory to prove the theorem.

Despite the difficulties presented by the student teachers in elaborating this proof, they were unanimous in defending the need to prove theorems in Basic Education, with the justification that they develop logical reasoning, thus fostering an understanding of the content, the interpretation of problem-situations and the search for strategies to solve them. The following statements confirm our claim. We believe that the meaning of demonstrations for the student teachers is the closest to what we call a formal proof.

> So, the idea of discussing demonstrations in the classroom is important, since the idea of presenting formulas upon formulas for them, in my opinion, is not teaching with meaningful learning. The presentation of demonstrations in basic education is important for student understanding, since most of the time, they describe problem solving. (Pre-service Teacher G)[4]
>
> I would discuss them first by showing the importance and use [of proofs] It is important to know where these things come from and why they work so coherently. Demonstrations also help develop logical reasoning in students. (Pre-service Teacher I)[5]

Based on the data analysis from the diagnostic investigation, we confirm our decision to elaborate a sequence using the Pythagorean Theorem. Initially, we considered a sequence using this topic due to the fact that it is a theorem suggested by recommended curricula and is present in all textbooks from the 8th and/or 9th years of Primary Education. Moreover, it is a theorem that has many applications in a range of contexts. Teachers acknowledge its importance and all student teachers studied it in Basic Education and applied it over the course of the Mathematics Teaching Degree Program.

Selecting, Organizing and Elaborating Learning Situations about the Pythagorean Theorem for the Training Process

To organize, elaborate and develop learning situations, we consider the assumption defended by Pietropaolo (2005): the teacher, at some point during their training, must experience the same situations that they will propose to their students, including situations that follow the sequence: "empirical verifications" => "semi-formal justifications" => "rigorous proofs", with the goal of elaborating activities that take this path, when appropriate. However, we also agree, with Pietropaolo, that this path is not necessarily linear: in the case of a student who already has experience with more formal demonstrations, this does not mean that, for new content, empirical verifications are no longer necessary. In other words, this path does not constitute stages of learning in students. It is possible, or rather, it is desirable, that at times the proposal of more empirical activities coincides with the proposal of formal ones.

For example, in the Learning Situation elaborated for studying Diophantine Equations, we do not believe it is appropriate to include a rigorous proof for Primary

[4] Original text: Então, o fato de discutir demonstrações na sala de aula é importante, pois o fato de colocar fórmulas e fórmulas para eles, em minha opinião não é um ensino com um aprendizado significativo. A apresentação das demonstrações no ensino básico é importante para o entendimento do aluno, pois na maioria dos casos descreve a resolução dos problemas.

[5] Original text: Discutiria mostrando primeiro a importância e utilidade [das provas]. É importante saber de onde as coisas saem e porque funcionam de forma tão coerente. Demonstrações também ajudam a desenvolver o raciocínio lógico dos alunos.

Education students. However, in the sequence involving the Pythagorean Theorem, we suggest to future teachers a sequence that takes into consideration the hierarchy discussed by Pietropaolo (2005) and we recommend that this be done with other content, when possible.

Our goal was to present and discuss with this group of students a path to teaching the Pythagorean Theorem that involves activities favoring the construction of a formal proof. We assume that this path, which includes experimentations, making conjectures, justifications, and empirical proofs towards the formal proof, broadens the conceptual image student-teachers have regarding teaching the metric relations of a right triangle. This path broadens the knowledge base of future teachers for teaching, particularly their Knowledge of Content and Teaching (KCT), according to Ball, Thames and Phelps (2008).

To elaborate our design, in relation to proof conceptions by students or teachers, we consider the works of Healy and Hoyles (2000), Dreyfus (2000) and Knuth (2002). Regarding the role of proofs in training Mathematics teachers, we use the works of Garnica (1995) and Pietropaolo (2005).

The sequence developed provided the group with discussions including incommensurable segments, the need to justify mathematically empirical verifications or "concrete" demonstrations, based on figures, to the formal development of a proof of the Pythagorean Theorem.

Activity 1 had two parts. The first part presented a historical approach and led the student to build an Egyptian square with ropes, using the Pythagorean triple 3, 4 and 5. The second part proposed a task to verify the Pythagorean relation by dividing the squares built with legs in unit squares and overlapping the square built on the hypotenuse with these unit squares. This was also done with the Pythagorean triple 3, 4 and 5, in order to understand the relation of the areas.

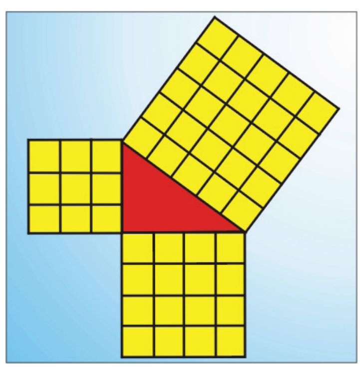


Figure 3. Activity 1: Pythagorean Theorem.

This activity was chosen because it is an activity frequently found in textbooks and, if proposed in isolation, it can lead students to forming incorrect concepts, such as a strong belief that one can always find a common unit in which a whole number of times fits in the sides of a triangle. In other words, the sides of the right triangles (segments) are commensurate pairs.

The goal of this activity was to promote a discussion on the possibility of doing this procedure for any right triangle whose leg measurements can be expressed by whole numbers. Based on this discussion, we saw the need to propose an activity in which it was possible to verify the validity of the Pythagorean relation for any right triangle. To this end, we proposed the Pythagorean game as Activity 2.

The game consists of dividing the squares built on the legs into pieces, such that one can always overlap the square built on the hypotenuse with these pieces, regardless of the measurements of the sides of the triangle.

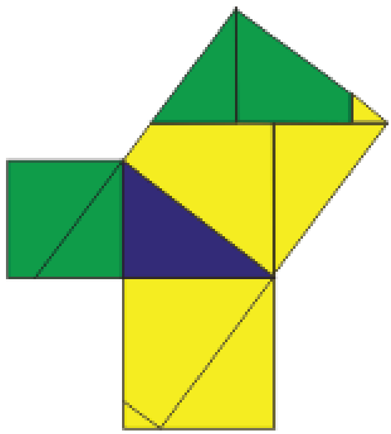


Figure 4. Activity 2: Pythagorean Theorem – Overlapping the square formed on the hypotenuse with the pieces of the squares built on the legs.

After placing the pieces in the square built on the hypotenuse, the student has to prove that this construction is valid. We expected the student teachers to indicate generally the measurements of all angles. Afterwards, they would link the complementary angles in the pieces built on the squares of the legs and prove that these pieces perfectly overlapped the square built on the hypotenuse.

In this activity, we must discuss the advantages of substituting the previously proposed activity 1 with this one; to analyze the degree of complexity and generalization of the proof involved; the need to prove that the pieces overlap the square even if they believed there was no doubt about it.

Since the student-teachers believed it was unnecessary to prove that the pieces of the puzzle in activity 2 overlapped the square formed on the hypotenuse, we proposed the activity “Building rectangles” or “64=65?” to demystify the idea that only the concrete composition/decomposition of a figure is enough to prove a mathematical conjecture involving areas.

The activity "64=65?" consists of taking a part of an 8x8 square in order to build a new "rectangle" with all the pieces whose sides measure 5 and 13 units of measure. If this construction were possible, it means 64=65. Therefore, the student would have to prove that what seems like a rectangle, in fact, is not.

The goal of this activity is to lead the student teachers to reflect on the role of empirical verification and the use of manipulatives to support the student in interpreting and understanding the problem, leading them to make conjectures. We believe it is important to propose activities that enable empirical verification and manipulation of concrete material, but associated to these activities, there should also be a discussion about the need to pursue mathematical arguments in order to validate, or invalidate, the conjectures they made, only by manipulating figures.

The purpose of proposing the list of questions to the student teachers was to promote a reflection on the goals that the aforementioned activity can develop in Basic Education.

Finally, we propose an activity whose goal is to arrive at the formal proof using manipulatives, which can subsequently foster the construction of the proof by students.

Using paper, a ruler, a square, a compass and a colored pencil, the students will build any three right triangles so that they are similar to one another. Overlapping the figures enables the students to recognize the existing relations between the sides. Having done this, the students will be instructed to build a table with the metric relations of the right triangle, identifying among them the relation known as the Pythagorean Theorem.

To complete the tasks we proposed, the future teachers should: experience the activities in a group like students from the 9th year of Primary Education; reflect, within each group, on the pedagogical potential of the activities and the possibilities to be implemented in the classroom; and elaborate an individual report at the end of each meeting, containing their reflections, including occasional disagreements in relation to other members of the group. Afterwards, the researchers/trainers proposed that each group share their discussions with the other students in the class, who then highlighted consensuses and possible divergences, thus organizing the discussions. This process allowed us to collect data for the study from audio recordings from group discussions, student logs while completing the activities and reports, and the field journals by the researchers.

We would like to point out that these students had little experience with teaching in practice (only in supervised activities offered by the program) and they almost always assumed the position of learners given the topics discussed, since they revealed a greater concern with learning the content than with reflecting on how to use it in their future practice. For this reason, we decided to analyze the data by grouping the categories by Ball, Thames and Phelps (2008) into pairs, since we believe that the analysis is richer in meaning due to the possible complementarity between the chosen pairs. Our decision to group the categories depended, evidently, on our perspective on the data. It is worth noting that other researchers could group the categories differently, if they considered it appropriate to do so.

We present the following pairs adopted: Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), Horizon Content Knowledge (KCS), Knowledge of Content and Students (KCT), Knowledge of Content and Teaching (HCK), Knowledge of Content and Curriculum (KCC).

Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK)

The student teachers correctly discussed the Pythagorean Theorem, thus showing Common Content Knowledge relative to this topic. However, when asked to present a formal proof, four of them were not able to outline even an idea. Based on this data, we confirmed our decision to develop learning situations that aim to broaden the future teachers' Specialized Content Knowledge of proofs, and consequently, of a theorem that everyone will inevitably teach when they are Basic Education teachers.

The statement by student teacher (E), for example, shows that, despite the proof for the Pythagorean Theorem being present in the curricula and in textbooks, it was knowledge that not all student teachers had and the learning situations proposed in our study enabled them to advance on this topic.

> The fact is that I was not familiar with these demonstrations (and it is worth noting that if you had asked me about a demonstration of the Pythagorean theorem before this course, I would not have known how to answer), but in general, I think about how interesting it is that there are, in fact, more than 400 demonstrations for a mathematical truth. (Pre-service Teacher E)[6]

We also believe that this sequence could provide future teachers with broadened Specialized Knowledge of Content, in the sense of being familiar with different ways of proving the theorem, and different arguments, so that they can make choices about how to address the content.

Student teachers and (H), when referring to the proofs suggested in our sequence for the Pythagorean Theorem, highlight the importance of being familiar with various approaches to the same content.

> I can see that all the activities presented new mathematical knowledge, because they showed us different ways to demonstrate the theorem, using manipulatives. (Pre-service Teacher C)[7]
>
> This week was very productive because I gained new knowledge. I learned different methods for demonstrating the Pythagorean Theorem, which I did not know before and this is very important for working with the Pythagorean Theorem in Basic Education in order to facilitate student learning. (Pre-service Teacher H)[8]

[6] Original text: É fato que não conhecia essas demonstrações (e vale ressaltar que se me perguntassem sobre alguma demonstração do Teorema de Pitágoras antes dessa disciplina, eu não saberia responder), mas, em geral, penso como é interessante o fato de que existam mais de 400 demonstrações para uma verdade matemática.

[7] Original text: Nota-se que todas as atividades trouxeram um conhecimento matemático novo, pois nos mostrou várias formas de demonstrar o teorema, usando material manipulável.

[8] Original text: Essa semana foi muito produtiva porque obtive novos conhecimentos, conheci diferentes métodos de

In the statement by student teacher (H), we observe not only the perspective of a student in the Mathematics Teaching Degree Program, concerned with learning more Mathematics, but also the perspective of a future teacher, who needs mathematical knowledge in order to reflect on their practice and make choices for teaching.

The statements by student teacher (J) reveal that, even though we have already worked with a familiar activity, we can revisit it from another perspective and include other arguments. In another instance, he addresses the idea of experimenting, of doing an activity that he previously saw others doing.

> I am already familiar with this activity [1], because it is a geometric demonstration presented in several books, but I love the part about the Egyptian triangle, which was new to me. (Pre-service Teacher J)
>
> It is worth mentioning that this activity offered me the opportunity to demonstrate the Pythagorean Theorem, because, although I have seen others do this demonstration, I have never stopped to try it. (Pre-service Teacher J)[9]

The purpose of activity 1, which includes a historical construction of a right triangle made by the Egyptians, was not only to motivate the students, but also to promote a discussion on the possibilities within the History of Mathematics as a context for teaching and learning concepts.

Moreover, by experiencing the learning situations as students, the student teachers were able to broaden their reflections on their future teaching practice, as shown by student teacher (E), during group discussions about the goals of some activities.

> The purpose of this activity is to spark student interest and enjoyment in mathematics, because the students themselves will see properties extend to real life that, up until that point, were only seen on the blackboard. (Pre-service Teacher E)[10]

Therefore, we believe that experiencing and reflecting on the activities from the Learning Situation helped broaden their knowledge base for teaching, Common Content Knowledge and Specialized Content Knowledge, domains by Ball, Thames and Phelps (2008).

However, it is fundamental to note that not all of the future teachers sufficiently increased their knowledge of incommensurable segments. We would like to reiterate that we encouraged reflections on this issue, although it was not the focus of this study, since

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demonstração do Teorema de Pitágoras que eu não conhecia e que é muito importante para trabalhar com o Teorema de Pitágoras na Educação Básica para facilitar a aprendizagem dos alunos.

[9] Original text: Eu já conhecia essa atividade [1], pois é a demonstração geométrica apresentada por diversos livros, mas adorei a parte do triângulo egípcio, foi algo novo para mim.

É bom ressaltar que esta atividade me oportunizou demonstrar o teorema de Pitágoras, pois apesar de já ter visto outros fazendo essa demonstração, nunca parei para tentar.

[10] Original text: Essa atividade tem como propósito despertar no aluno o interesse e o prazer pela matemática, pois os próprios alunos verão propriedades até então vistas no plano da lousa se estenderem para a vida real.

we determined that not all of the subjects in our study had the fundamental concepts related to the Pythagorean Theorem, such as the meaning of incommensurable segments.

Perhaps for this reason, when discussing activity 1, the student teachers considered it an excellent activity to start teaching the Pythagorean Theorem. However, this judgment by the students is incorrect, because it is valid only for specific cases and not for the triangles whose legs are incommensurable relative to the hypotenuse. In other words, these students did not take into consideration the issue of incommensurability of the segments. The following statement shows that the issue of incommensurability was not even considered because the student teachers did not have this Knowledge before our Training process.

> From a mathematical point of view, the question posed about the validity of the activity for any real value in the sides of the triangle presented concepts that I was not familiar with. (Pre-service Teacher E)[11]

Incommensurable segments is a topic that should be developed in Basic Education, according to curricula such as the National Curricular Guidelines (PCN, 1998). For this reason, it should be part of the teacher's Common Knowledge.

Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT)

Regarding pedagogical knowledge for teaching, we will discuss Knowledge of Content and Teaching together with Knowledge of Content and Students, since we believe that these two categories, in this case, are closely related.

In the reflections made by the student teachers on the learning situations, we observed an appreciation of activities in which the students could make conjectures and build their own knowledge.

> From the pedagogical point of view, [this activity] provided me with a new tool that makes students build their own knowledge. (Pre-service Teacher D)[12]
>
> [...] we see that this activity provides students with opportunities to make discoveries, without something being imposed without addressing the material. (Pre-service Teacher E)[13]

Creating knowledge is one of the aspects identified by Knuth (2002) when investigating the role that proofs can play when incorporated into Mathematics classes.

[11] Original text: Do ponto de vista matemático, o questionamento feito sobre a validade da atividade para qualquer valor real nos lados do triângulo mostrou conceitos que eu não conhecia.

[12] Original text: Do ponto de vista pedagógico [essa atividade] me forneceu uma nova ferramenta que faça com que os alunos construam seu próprio conhecimento.

[13] Original text: [...] é observado nessa atividade, um caráter viabilizador em fornecer oportunidades de descobertas aos alunos, sem que alguma coisa seja imposta sem o tratamento do material.

This construction of knowledge can take place in different ways. The student teachers believe, for example, that the use of manipulatives can promote learning and can even be used in activities that have the goal of building a formal proof. We highlight the following statements, regarding the elaboration of a formal proof based on an activity of building three similar right triangles.

> This activity is a good starting point for a formal demonstration, because with the material, it is easier to see the existing relations. (Pre-service Teacher H)[14]
>
> This activity would be very interesting to apply in Basic Education, because the students can more easily perceive the similarity between triangles, when a (triangle) is overlapping another. (Pre-service Teacher A)[15]
>
> These activities can also lead students to reflect on doing mathematics and the nature of mathematical knowledge. (Pre-service Teacher B)[16]

Student teacher (F), for example, analyzes the advantages of starting a formal proof of the Pythagorean Theorem with the activity in which students can overlap right triangles to better observe their similarity and deduce the existing relations.

> [...] I reflected on this activity and I noticed that it could be used to start a formal demonstration of the Pythagorean Theorem. It can provide the student with a better understanding of the theorem, in addition to sharpening their geometric knowledge. (Pre-service Teacher F)[17]

Student teachers were able to progress towards evaluating the advantages of using a determined activity, making choices for teaching and justifying them, as we can see in the following statement.

> This game [Activity 2] presented us with some different reflections from the previous activity. After finishing this game, I would not start the demonstration with the previous activity. I would start with this one, because it seemed more complete, despite also being more difficult. (Pre-service Teacher I)[18]

[14] Original text: Essa atividade é um bom ponto de partida para demonstração formal, pois com o material fica mais fácil ver as relações existentes.

[15] Original text: A atividade é bastante interessante para ser aplicada na Educação Básica, pois com ela os alunos poderão perceber mais facilmente a semelhança entre triângulos, quando colocarem um (triângulo) sobreposto ao outro.

[16] Original text: Tais atividades ainda podem levar os alunos a refletirem sobre o fazer matemática e a questão da natureza do conhecimento matemático.

[17] Original text: [...] fiz uma reflexão sobre essa atividade e pude notar que ela pode ser utilizada para dar início a demonstração formal do Teorema de Pitágoras. Pois, ela pode possibilitar ao aluno uma melhor compreensão sobre o teorema, além de aguçar o conhecimento geométrico do mesmo.

[18] Original text: Este jogo [Atividade 2] nos trouxe algumas reflexões distintas da atividade anterior, após o término deste jogo eu não começaria a demonstração com a atividade anterior, eu começaria com esta, pois me pareceu mais completa, apesar de mais difícil também.

Organizing instructions, selecting and elaborating activities, and preparing teaching materials, in order to build a sequence that enhances learning a topic, is part of the knowledge necessary for the teacher to teach. We observed that the student teachers preferred to use less rigorous proofs, in order to foster an understanding of the Theorem and introduce the formal proof only afterwards.

The proposed situations made the students discuss the feasibility of using these activities with Basic Education students, taking into consideration the need to go beyond empirical verifications and providing arguments that promote generalization. When student teacher (D) analyzed the activity "64=65?", he pointed out that

> [...] this activity offers us an argument, or a tool to show students that we cannot trust manipulatives on their own. (Pre-service Teacher D)[19]

Meanwhile, in addition to analyzing the generalizability of the proofs, student (C) is concerned with the impact that an activity can have on students in developing misconceptions on a topic, because it does not take into account a discussion of the cases for which it is valid.

> [...] we can see that the activities completed up to that point have shown us that we often draw conclusions without even seeing if they are truly valid for any case. This makes us open our eyes and pay more attention to what we are doing and how we plan for our students. (Pre-service Teacher C)[20]

We believe that, in this case, there is a strong relation between Knowledge of Content and Teaching and Knowledge of Content and Students, because in order to organize one's instruction, teachers should have in-depth knowledge about their target audience. In many statements, there is an intersection between these categories. However, since we are dealing with student-teachers, their reflections on students does not include classroom experience, but a few impressions from work done in the school in teacher training activities, whether in a supervised internship, or in projects with this purpose.

Moreover, due to their experience as students, the student teachers have in themselves an image of what they will find in school when they are Basic Education teachers and they speculate about the difficulties students will face, based on their own experiences. In addition, it is from this perspective that they analyze, for example, the

[19] Original text: [...] esta atividade nos oferece um argumento, ou ferramenta para mostrar aos alunos que não podemos confiar somente no material manipulável.

[20] Original text: [...] podemos notar que as atividades feitas até então nos mostram que muitas vezes nós fazemos nossas conclusões sem mesmo olhar se realmente estão sendo válidas para qualquer caso. Isso nos faz abrir os olhos e prestar mais atenção no que fazemos e planejamos para nossos alunos.

potential difficulties students will have when doing a certain activity, or its possible applications, as we can see in the following statement.

> For Basic Education, the "metric relations in right triangles" activity is very important before doing a demonstration of the theorem, because, with it, the students will better understand the notion of similarity between right triangles... While the demonstration is simple, it requires a notion of similarity that the students may not have. (Pre-service Teacher A)[21]

However, we believe that this sequence fostered group discussions on the teaching practice, the act of planning what to teach, the possibilities of using an activity with students in Primary Education, thus broadening their Knowledge of Content and Teaching and, despite being an initial training context, their Knowledge of Content and Students as well.

Horizon Content Knowledge (HCK) and Knowledge of Content and Curriculum (KCC)

When we proposed activities to determine and broaden the knowledge of future teachers on proofs and demonstrations, another goal of ours was to discuss the curriculum and the relations within the content, from the point of view of official documents (KCC), as well as from the point of view of the teacher and the way the curriculum is addressed in the classroom context (HCK).

Among the activity's goals, student-teacher (D) highlights reflection on the proof as a transversal topic, which permeates all mathematics teaching and is necessary in order to better develop other activities, whether in mathematics, other disciplines, or in everyday life. It also brings to the discussion an aspect identified by Knuth (2002), the development of logical thinking.

> It is not always the case that what "seems to be, is"; we have to investigate and, if possible, verify that the sentence is true. This way, students can reinforce their investigative side and develop their logical reasoning. [...] we cannot forget that this strengthens students' geometric perspective and how they organize their ideas in a demonstration. (Pre-service Teacher D)[22]

The "64=65?" activity was also useful for us to discuss the use of figures (manipulatives) to support a proof. This activity is not tied to a single topic, but

[21] Original text: Para a Educação Básica a atividade "relações métricas nos triângulos retângulos" é muito importante antes de fazer a demonstração do teorema, pois com ela os alunos entenderão melhor a noção de semelhança entre triângulos retângulos... Pois, por mais que a demonstração seja simples, exige uma noção de semelhança que os alunos podem não ter.

[22] Original text: Que nem sempre o que "parece ser, é", temos que investigar e se possível verificar que a sentença é verdadeira. Com isso, o aluno poderá reforçar seu aspecto investigativo e desenvolver seu raciocínio lógico. [...] não podermos esquecer que fortalece a visão geométrica do aluno e a organização das ideias numa demonstração.

to the idea of formalization, the adequate use of language and the rules specific to Mathematics.

> With this activity, we can achieve some goals, such as... understanding that the figure alone is not enough for us to make all kinds of claims that we can get by observing it. (Student teacher A)[23]

Our goal with this activity was not to recommend using manipulatives or any other activity that uses empirical verification. This type of activity encourages making conjectures and finding conclusions based on observation. We defend the use of these activities, but our purpose was for the student teachers to reflect on the possibilities of advancing beyond them. In addition, this goal was accomplished, as we can see in the following statements:

> To show them that not everything is what it seems; however, after this inductive idea in geometry, we can introduce the importance of a formal demonstration. (Pre-service Teacher I)[24]
>
> These activities can even lead students to reflect on doing mathematics and the nature of mathematical knowledge, that is, because our feelings, as well as perspective, for example, are often not enough for claiming mathematical truths; there must be an algebraic proof. (Pre-service Teacher B)[25]

We believe that the "64=65?" activity, despite being very well-known, facilitates a rich discussion in this context on planning a sequence that promotes knowledge-building by the student, and on the limitations and potential of this kind of activity, which can evoke in students the need to improve their arguments in order to convince themselves and their colleagues.

Student-teachers (C) and (D), as we can see in the following protocols, associate other topics to the Pythagorean theorem and indicate the goal of developing geometric thought, which is defended in curricula and official documents as a way to promote observation and an understanding of relations and problem-solving.

> The goal of this activity is to demonstrate the Pythagorean Theorem, but, to achieve this final goal, we find many others, such as to develop the student's perspective towards geometry... (Pre-service Teacher C)[26]

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[23] Original text: Com a atividade podemos atingir alguns objetivos, como... compreender que só a figura não é bastante para fazermos todos os tipos de afirmação que podemos obter observando-a.
[24] Original text: Mostrar para eles que nem tudo que parece é, porém após essa ideia indutiva de geometria, introduziremos a importância de uma demonstração formal.
[25] Original text: Tais atividades ainda podem levar os alunos a refletirem sobre o fazer Matemática e a questão da natureza do conhecimento matemático, isto é, porque muitas vezes nossos sentidos, como a visão, por exemplo, não bastam para afirmar verdades matemáticas, é necessário que haja uma prova algébrica.
[26] Original text: A atividade tem por objetivo demonstrar o Teorema de Pitágoras, mas para chegar nesse objetivo final caímos em outros tantos, como desenvolver o olhar do aluno para a visão geométrica...

> [...] by proving the veracity of the Pythagorean relation, aside from building and rebuilding their concepts, they will reinforce their concepts about angles and areas, in addition to developing their geometric perspective. (Pre-service Teacher D)[27]

On the other hand, despite the student teachers associating the proofs of the Pythagorean Theorem to other content, the idea of a prerequisite is still very present in the discourse, as well as little concern with highlighting key points of a determined topic, in order to benefit learning future content. The statements by student teachers (F) and (I) justify our analysis.

> I think this activity is potentially rich for studying in basic education, because it is shown to be very effective in demonstrating the theorem. However, the student must already have some prerequisites on subjects, such as similarity of triangles, relations of congruence (side, angle). So that they can really understand what they are doing. (Pre-service Teacher F)[28]
>
> This way, they remember some properties on equivalent areas, congruent angles, as well as the relation of complementary and supplementary angles and, at the same time, they prove a very important result in Mathematics. (Pre-service Teacher I)[29]

However, student teacher (A) analyzes the activity "overlapping triangles", whose goal is to list the existing relations in the similar triangles, in a different way, anticipating the formal proof and emphasizing aspects that will benefit student understanding in an activity that is still to come.

> This activity is a good starting point, because when the teacher does a formal demonstration of the Pythagorean Theorem on the board, the students will understand similarities of triangles more quickly. (Pre-service Teacher A)[30]

What to teach and how to teach is still very much associated to the prerequisites they believe are necessary. It seems to us that the student teachers cannot conceive of seeking strategies to teach students without this knowledge.

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[27] Original text: [...] comprovar a veracidade da relação de Pitágoras além do que eles terão que construir e reconstruir seus conceitos, reforçarão seus conceitos sobre ângulos e áreas, além de desenvolverem sua visão geométrica.

[28] Original text: Considero assim essa atividade potencialmente rica, para ser trabalhada na educação básica. Pois ela se mostra bastante eficaz na demonstração do teorema. Porém, é necessário que o aluno já tenha alguns pré-requisitos sobre assuntos, como: semelhança de triângulos, relações de congruências (lado, ângulo). Para que eles realmente compreendam o que estão realizando.

[29] Original text: Assim eles recordam algumas propriedades de equivalência de área, congruência de ângulos, assim como a relação de ângulos complementares e suplementares, e ao mesmo tempo provam um resultado importantíssimo na Matemática.

[30] Original text: A atividade é um bom ponto de partida, pois quando o professor for fazer a demonstração formal do Teorema de Pitágoras no quadro os alunos compreenderão mais rápido as semelhanças dos triângulos.

CONCLUSIONS

The data collected in the diagnostic phase show that the student teachers believe it is important to work on proofs of the Pythagorean Theorem in Basic Education classes, though many of them were not able to outline a single proof for this theorem. Others, in an attempt to present a proof for the theorem, made use of theory over the course of the construction, which reveals a conceptual error about what a proof means.

The intention of the learning situations that were applied over the course of our training, and constantly reworked, was for the group to advance beyond the difficulties observed in the diagnostic phase. The set of data obtained in the first phase, together with the reflections by the student teachers on the activities included in the learning situations, enabled us to answer the second research question.

The discussions resulting from the learning situations broadened the knowledge base necessary for Mathematics teachers to exercise their profession and gave new meaning to working with proofs, considering the change in conception on the meaning of proofs.

We observed a certain cautiousness and doubt, on the part of the students, to indicate activities for Primary Education students. For most of the student teachers, the idea remained that not all students would be able to develop skills relative to proofs using formal language. However, with the expanded meaning of proofs, the future teachers began to believe that more empirical and less formal proofs are necessary for students to know why a claim is true, that is, in order to defend proofs with an explanatory purpose.

After the intervention, with respect to the knowledge necessary for the teacher to teach the content using activities that require proofs, it is important to note that the sequences and discussions we encouraged contributed to broadening not only their knowledge of content and curriculum, but also an understanding of strategies that can benefit building concepts and attitudes by including proofs in Basic Education Mathematics classes.

Finally, the future teachers started to assume a broader sense of proofs in Basic Education Mathematics classes: according to them, it is not enough for the student or teacher to simply reproduce the proofs present in books, but they must also do mathematics, which includes experimentations, justifications, conjectures and, when applicable, rigorous proofs.

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DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the corresponding author, M.E.A., upon reasonable request by e-mail.

REFERENCES

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), p.389-407.

Brasil (1996). *Lei nº 9.394 de 20 de dezembro de 1996*. Das diretrizes e bases da educação nacional. Diário Oficial da União. Brasília.

Brasil (1998). Secretaria de Educação Fundamental. *Parâmetros Curriculares Nacionais (5ª a 8ª séries)*. Brasília: MEC.

Healy, L. & Hoyles, C. (2000). A study of proof conceptions in algebra. Journal for Research in Mathematics Education, 31(4), p.396-428.

Knuth, E. (2002). Teachers' conceptions of proof in the context of Secondary School Mathematics. Journal of Mathematics Teacher Education, 5(1), p.61-88.

Mateus, M. E. A. (2015). *Um estudo sobre os conhecimentos necessários ao professor de Matemática para a exploração de noções concernentes às demonstrações e provas na Educação Básica.* 267f. Tese (Doutorado em Educação Matemática) – Universidade Anhanguera de São Paulo.

Pietropaolo, R. C. (2005). *(Re)Significar a demonstração nos currículos da educação básica e da formação de professores de matemática.* 249f. Tese (Doutorado em Educação Matemática) – Pontifícia Universidade Católica, São Paulo.

SBEM (2003). Sociedade Brasileira de Educação Matemática. *Subsídios para a discussão de propostas para os cursos de Licenciatura em Matemática: uma contribuição da Sociedade Brasileira de Educação Matemática*. São Paulo: SBEM.