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Epistemic and Cognitive Analysis of Proportionality Tasks from the Algebraization Levels Perspective

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ABSTRACT

In this article, we present the results of the evaluation phase of a training intervention with primary education prospective teachers on the subject of proportionality. The focus is on developing the epistemic and cognitive analysis competence of mathematical practices in problem solving. The prospective teachers first solve a mathematical task of inverse proportionality in several ways, then identify the mathematical practices employed in the resolutions, the objects and meanings, the difficulties, the levels of algebrization involved, and finally, perform the analysis of two solutions proposed by school pupils to a problem of direct proportionality. The results indicate that the future teachers' knowledge about proportionality present deficiencies that can hinder the teaching of the subject. Likewise, it is concluded that developing the required professional knowledge and skills requires the application of longer training interventions.

Keywords: Proportionality, teacher training, onto-semiotic approach, algebraization levels, epistemic and cognitive analysis.

Análisis Epistémico y Cognitivo de Tareas de Proporcionalidad desde la Perspectiva de los Niveles de Algebrización

RESUMEN

En este artículo se presentan los resultados de la fase de evaluación de una intervención formativa con futuros maestros de Educación Primaria sobre el tema de proporcionalidad. Se centra la atención en el desarrollo de la competencia de análisis epistémico y cognitivo de las prácticas matemáticas en la resolución de problemas. Los futuros maestros resuelven primero una tarea matemática de proporcionalidad inversa de varias maneras, luego identifican las prácticas matemáticas empleadas en las resoluciones, los objetos y significados involucrados, las dificultades implicadas, los niveles de algebrización puestos en juego, y finalmente, realizan el análisis de dos soluciones propuestas por alumnos de primaria a un problema de proporcionalidad directa. Los

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resultados indican que los conocimientos sobre proporcionalidad de los futuros profesores presentan deficiencias que pueden dificultar la enseñanza del tema. Asimismo, se concluye que el desarrollo de los conocimientos y competencias profesionales pretendidos requiera aplicar intervenciones formativas de mayor duración.

Palabras claves: Proporcionalidad, formación de profesores, enfoque ontosemiótico, niveles de algebrización, análisis epistémico y cognitivo.

Análise Epistêmica e Cognitiva de Tarefas de Proporcionalidade na Perspectiva dos Níveis de Algebrização

RESUMO

Este artigo apresenta os resultados da fase de avaliação de uma intervenção de formação com futuros professores do ensino primário sobre a questão da proporcionalidade. A atenção está centrada no desenvolvimento da competência na análise epistémica e cognitiva das práticas matemáticas na resolução de problemas. Futuros professores primeiro resolver uma tarefa matemática de proporcionalidade inversa de várias maneiras, em seguida, identificar as práticas matemáticas utilizadas nas resoluções, os objetos e significados envolvidos, as dificuldades envolvidas, os níveis de algebrização em jogo e, finalmente, realizar a análise de duas soluções propostas pelos alunos da escola primária para um problema de proporcionalidade direta. Os resultados indicam que o conhecimento sobre a proporcionalidade dos futuros professores apresenta deficiências que podem dificultar o ensino da disciplina. Da mesma forma, conclui-se que o desenvolvimento do conhecimento e das competências profissionais requer a aplicação de intervenções de formação mais duradouras.

Palavras-chave: Proporcionalidade, formação de professores, abordagem ontosemiótica, níveis de algebrização, análise epistêmica e cognitiva.

INTRODUCTION

Developing the students' mathematical knowledge and competencies is associated with their teachers' didactic-mathematical education. Consequently, in the field of research in mathematics education, a clear concern to determine the type of didactic-mathematical knowledge required by the mathematics teacher to develop a suitable teaching is observed (Chapman, 2014; Ball, 2009; Sowder, 2007).

From the perspective of the onto-semiotic approach to mathematical knowledge and instruction (OSA) (Godino, Batanero & Font, 2007), the activity of epistemic and cognitive analysis, as well as the recognition of algebrization levels in mathematical problem-solving tasks, are considered as ways to develop that knowledge (Burgos, Beltrán-Pellicer, Giacomone & Godino, 2018, Godino, Aké, Gonzato & Wilhelmi, 2014, Rivas, Godino & Castro, 2012).

With regard to the study of proportionality in the field of teacher education, there is a growing development of research focused on the study of mathematical knowledge *per se* and the pedagogical knowledge required to teach proportionality (Izsák & Jacobson, 2013; Sowder et al, 1998). As various works of research show, both teachers in initial and in-service training have difficulties in teaching concepts related to proportionality (Ben-Chaim, Keret & Ilany, 2012, Berk, Taber, Gorowara & Poetzl, 2009, Buforn, Llinares & Fernández, 2018; Rivas, Godino & Castro, 2012). The difficulties that teachers have in the primary and secondary stages with the concepts of ratio and proportion, motivate the use of algorithmic procedures such as the rule of three to teach how to solve problems that require proportional reasoning.

In this order of ideas, from the perspective of the OSA referred to above, training actions have been developed, aimed at analysing the initial knowledge and evaluating the degree of development of the epistemic analysis competence to solve proportionality tasks (Burgos, Giacomone, Beltrán-Pellicer & Godino, 2017, Burgos et al., 2018). In general, the results of these investigations show that future teachers present deficiencies in didactic-mathematical knowledge and a deficient and biased conception on the nature of the elementary algebraic reasoning involved in proportionality tasks. In particular, the prospective teachers' difficulties to recognise the practices, objects and processes involved showed the need to do further research into the design and experimentation of new training interventions.

In this order of ideas, the research problem posed in the article is to design, implement and evaluate a training intervention with future primary education teachers aimed at:

- developing didactic-mathematical knowledge and skills on proportional reasoning and its relationship with the algebraic reasoning;
- encouraging the competence of epistemic-cognitive analysis of objects and processes put at stake in mathematical practices involved in solving proportionality tasks.

As mentioned earlier, this article reports on the evaluation results of a specific training intervention in that direction. Consequently, the following research question is posed: What aspects of didactic-mathematical knowledge and competence on proportionality and elementary algebraic reasoning can be observed/ developed by means of a training intervention that includes the use of the analysis tools proposed by the onto-semiotic approach?

In this article, the focus is on evaluating the advanced and specialized knowledge, as well as the competence in epistemic-cognitive analysis achieved by future teachers, when the proposed tasks involve the notion of proportionality and the use of different levels of algebrization.

The article is organized in the following sections. In the second section, the theoretical elements of the onto-semiotic approach are introduced. The third section describes the method used, including the context, the participants and the data collection and analysis instrument. The results of the analysis of the tasks are presented in the fourth section. The fifth section discusses the results obtained by identifying the level of competence of epistemic and cognitive analysis of future teachers.

THEORETICAL FRAMEWORK

The theoretical framework that we use in this article is the Onto-semiotic Approach to Mathematical Knowledge and Instruction (OSA) developed by Godino and collaborators (Godino, Batanero & Font, 2007). To follow we detail the Didactic-Mathematical Knowledge and Competence model (DMKC), and the model of algebrization levels developed in the OSA that constitute the theoretical tools with which the research problem will be addressed.

The Didactic-Mathematical Knowledge and Competence Model

The DMKC model of knowledge and competence of the mathematics teacher proposed in Godino, Giacomone, Batanero and Font (2017), develops the model of Didactic-Mathematical Knowledge described in Godino (2009). In the DMKC model, the categories of knowledge and didactic competencies of the mathematics teacher are articulated through the facets and components of a mathematical instructional process defined in the OSA.

According to the DMKC model, the teacher, in addition to having a *common mathematical knowledge*, corresponding to the educational level where he/she teaches, and an *extended mathematical knowledge* of the content, which allows him/her to teach in the higher stage, should have a *specialized content knowledge*. This specialized knowledge refers to the different facets involved in the mathematics teaching and learning process (epistemic, ecological, cognitive, affective, instructional and mediational facets). On the other hand, the teacher should be competent to determine the configurations of objects and mathematical processes involved in the practices put at stake in the intended meanings of the contents (epistemic configurations) as well as the configurations that the students put into play when solving the problems (cognitive configurations) (Godino, Giacomone, et al., 2017).

Algebraization Levels

Godino et al. (2014) propose a model of algebraic reasoning for Primary Education where criteria to distinguish purely arithmetic mathematical activity (level 0 of algebrization) from progressive levels of algebrization are established. The criteria to delimit the different levels are based on:

1) Type of objects: concepts (mathematical entities that can be introduced by descriptions or definitions), propositions (properties or attributes, statements about concepts), procedures (calculation techniques, operations and algorithms), arguments (statements required to justify the propositions or explain the procedures).
2) Type of representations used (languages in their different registers).

3) Generalization processes involved.

4) Analytical calculation put into play in the corresponding mathematical activity.

Associated with the algebrization levels, Godino, Beltrán-Pellicer, Burgos and Giacomone (2017) consider three specific pragmatic meanings of proportionality that are activated in the solution of tasks involving the proportionality of magnitudes: arithmetic, proto-algebraic and algebraic-functional meanings. The arithmetic meaning (Level 0 of algebrization) is characterized by the application of arithmetic calculation procedures (multiplication, division). In practice, particular numerical values intervene and arithmetic operations are applied on the aforementioned values; no objects and algebraic processes intervene. The Proto-algebraic meaning is focused on the notion of proportion, so that the recognition of the unit value in a unit reduction procedure, and the use of diagrammatic representations of solutions can be described as Proto-algebraic Level 1. On the other hand, the solution of a missing value problem, based on the use of ratios and proportions, involves an unknown and the writing of an equation. The algebrization activity performed in this case is Proto-algebraic Level 2, since the unknown is in a single member of the equation established ($Ax = B$).

Teachers' recognition of the different algebrization levels when solving mathematical tasks, in particular, in those situations that put into play the notion of proportionality, is considered a key aspect of the DMKC model on this content (Burgos et al. al., 2018; Godino, Beltrán-Pellicer et al., 2017). This is because the recognition of objects and processes characteristic of elementary algebraic reasoning allows the identification of progressive stages of proportional reasoning.

METHODOLOGY

Methodological Approach

Taking into account the research problem, the methodological framework will be didactic engineering, understood in the generalized sense proposed by the OSA (Godino, Rivas, Arteaga, Lasa and Wilhelmi, 2014). This interpretation extends its traditional conception (Artigue, 1989) in the direction of research-based design (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Hence, it distinguishes four research phases: 1) preliminary study (considering the epistemic-ecological, cognitive-affective and instructional facets); 2) design of the experiment (selection of tasks, sequencing and *a priori* analysis of them according to the expected students' behaviour); 2) implementation (observation of the interactions between people and resources and assessment of the lessons learned); 4) evaluation (retrospective analysis derived from the contrast between what is foreseen in the design and what is observed in the implementation).

Research Context and Participants

The training experience was carried out in two sessions with a group of 35 future teachers in the third year of the Primary Education Degree. In the first session a 2-hour workshop was held, in which the characteristics of the Elementary Algebraic Reasoning (EAR) and the model of algebrization levels were presented, making use of proportionality tasks.

In the following session (2 hours), two tasks of proportionality (one focused on the concept of sharing and ratio and the other of inverse proportionality) were proposed to the students. Working in teams, the students had to: (a) solve the tasks in several ways, including strategies that could be used by primary school pupils to solve the problem; (b) identify the knowledge put into play in the various solutions, enumerating the sequence of practices carried out to solve the problem; (c) discuss the different strategies used to solve the problem, identifying the difficulties that may arise in solving the problem using each strategy; and (d) assign levels of algebraic reasoning to the different strategies, taking into account the algebraic objects and processes previously identified.

As an optional complementary work, to increase the final grade of the course, the task described in the following section was proposed. The results of this research only refer to what was obtained by means of the optional work, which was individually carried out by 12 future teachers, after the completion of the course, revealing relevant aspects of the learning achieved by them.

Data Collection Instrument

In the first place, it is important to identify whether future teachers correctly distinguish and solve situations of inverse proportionality, which constitutes an aspect of the extended content knowledge, given that inverse proportionality is not included in primary education. Secondly, to evaluate aspects of the epistemic and cognitive facets, didactic-mathematical knowledge is intended, namely:

- the flexibility to solve a problem using different resolution strategies (epistemic facet);
- to identify levels of algebraic reasoning (epistemic facet);
- to recognize the difficulties that pupils may encounter (cognitive facet);
- to analyse answers given by primary school pupils (cognitive facet).

Below are the questions posed to the students in the optional work:

1. Solve in several ways the following mathematical task. Use all the strategies you know, including the strategies you think your pupils would use to solve the problem.

It's the graduation party at the Las Gaviotas Institute. 7 students have been chosen to design and decorate the auditorium. The 7 would need to work 21 hours to leave the room ready for the celebration. Unfortunately, before they could start with the task, 4 guys have become sick with *chickenpox and have to stay home. How many hours will it take the remaining students to design and decorate the room? Describe and explain the strategy you used to give your answer.*

2. For each solution, list the sequence of practices that are carried out to solve the problem and complete the table included below, adding the necessary rows.

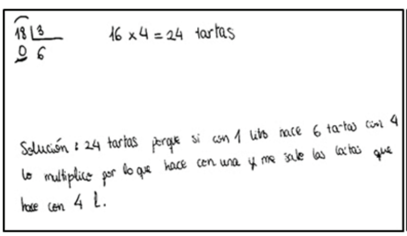
Sequence of elementary practices to solve the problem	*Objects referred to in the practices (concepts, propositions, procedures, arguments.)*

3. Detail what difficulties you can observe in solving the problem using each strategy (to do so, observe the practices, objects and processes identified as potentially conflicting for the pupils).

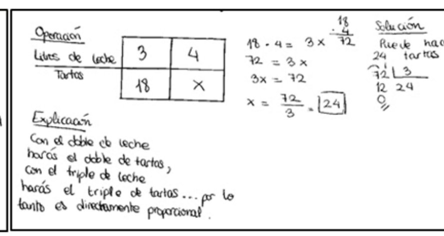
4. Assign levels of algebraic reasoning to the different solutions given in the previous items to the task, taking into account the objects and algebraic processes previously identified.

5. Next, the solutions given by two children to the following problem appear:

A baker uses 3 litres of milk to make 18 equal cakes. How many cakes can you make with 4 litres of milk? Explain how you found out the answer.



Pupil 1



Pupil 2

Solution: 24 cakes because if with 1 litre he makes 6 cakes with 4 I multiply it by what he does with one and I get the cakes he makes with 4 l.	Explanation. With double milk, you will make twice as many cakes, with three times as much milk you will make three times as many cakes ... therefore it is directly proportional.

a) Do you think the answers (resolution and argumentation) given by the pupils are correct?

b) What level of algebrization do you assign to the different answers? Justify your answer.

c) Identify in the solutions given by the pupils the different elementary practices involved (steps taken in the resolution of the problem).

The data treated in this article, collected through the questions referred to, come from a common and optional activity used by teacher educators in initial teacher training, for which the approval of the Ethics Committee for its collection and analysis has not been required.

RESULTS AND ANALYSIS

Solution Strategies According to Algebrization Levels

All students solved the task, ten of them in at least two different ways. The following is a description of the students' solution strategies, classified according to their algebrization levels:

– *Level 0:* Arithmetic. Student S6 shows an example of the use of this strategy (Figure 1). S6 operates on particular numbers, in natural and numerical language. No algebraic objects and processes were involved.

21 hours × 7 students = 147 hours in total.

7 students – 4 sick students = 3 students remain.

21 hours × 4 students = 84 hours.

147 total hours – 84 hours of the 4 students = 63 hours must be performed between the 3 students.

84 hours of the 4 sick students: 3 remaining students = 28 hours each of the 3 students.

28 hours of each of the three students + 21 who had initially = 49 hours must be performed by each of the three students.

Figure 1. Algebrization Level 0 (Arithmetic) response, given by S6.

– *Level 1*: Reduction to the unit. The students who used this strategy used the tabular register. An example of this strategy can be seen in Figure 2.

The recognition of the unit value (number of hours that a single person would use) involved in the procedure of reduction to the unit, and the use of diagrammatic-tabular representations in the solution are qualified as Proto-algebraic of Level 1. It is observed that the student S11 (Figure 2) recognizes properties of the inverse proportionality relationship represented also through the tabular register (if the quantities of one magnitude are multiplied by a number, the corresponding quantities of the other are divided by the same number).

You have to start with the knowledge that fewer people means more time. In order to get the solution we first find out how long it would take a person, taking into account that when you reduce from 7 to the unit you divide by 7, and the hours are multiplied by 7, being inversely proportional

People	*Hours*
7	*21*
7/7=1	*21x7=147*

As we already have the unity, we pass it to 3 people, and taking into account that when multiplying now in people, we have to divide in hours, being inversely proportional:

People	*Hours*
7	*21*
7/7=1	*21x7=147*
1x3=3	*147/3=49*

Figure 2. Proto-algebraic Algebrization Level 1 response, given by S11.

– *Level 2:* Rule of three/Proportional Equation. The solution of the problem by means of the rule of three involves an unknown and solving an equation in which the unknown is in only one element of the equation. In this sense, the activity developed is considered as Proto-algebraic of Level 2. Student S2 presents an example of the use of this strategy (Figure 3).

We are proposing a rule of three, which will be solved in the reverse way because we know that when there are fewer children, it will take more hours.

$$\left.\begin{array}{l} 7 \rightarrow 21 \\ 3 \rightarrow x \end{array}\right\} 3x = 21 \cdot 7; x = \frac{147}{3}; x = 49$$

Figure 3. Proto-algebraic Algebrization Level 2 response, given by S2.

Table 1 shows the frequencies of correct and incorrect responses grouped according to the algebrization levels exemplified above.

Seven of 12 students proposed solutions that were always incorrect and two offered a correct and an incorrect solution to the problem. The most used strategy was the rule of three, followed by the arithmetic type, being in both cases the number of incorrect answers considerably greater than the number of correct answers.

Table 1
Frequency of responses according to algebrization level and correction degree.

Algebrization level	Correct	Incorrect	Total according to level
Level 0	2	7	9
Level 1	3	1	4
Level 2	4	7	11
Total (*N* = 12)	9	15	24

In students' reports, it is observed that the wrong answers are fundamentally due to two factors:

1. *To consider that the relationship is one of direct proportionality*. Of the seven students who solved the problem incorrectly, five did it so because they assumed that the number of children and the number of hours of work were directly proportional magnitudes. An example of this response, given by student S4, is shown in Figure 4.

$$\left.\begin{array}{l} 7\ students \rightarrow 21\ hours \\ 3\ students \rightarrow x\ hours \end{array}\right\} x = \frac{3 \cdot 21}{7} x{=}9\ hours\ each\ child$$
$$9\ hours \times 3 child{=}27\ hours\ total.$$

Figure 4. Incorrect solution using the rule of three assuming a relationship of direct proportionality. Response given by S4.

2. *To interpret 21 hours as the total number of hours needed to design and decorate the room*. Of the seven students who solved the problem incorrectly, one made this mistake (student S12) (Figure 5). In addition, two students who had offered other correct solutions presented this error. Three students made this error.

SOLUTION 3 (inverse proportionality, rule of 3)

1. The 7 students would need to work 21 hours to leave the room ready for the celebration.

2. If we divide the total hours among the initially existing students, we know that each student would work 3 hours (21/7 = 3 hours).

3. Then, faced with the illness of 4 of them, we have to find out how many students will be able to design and decorate the assembly hall: 7-4 = 3 students

4. In inverse proportionality. The smaller the number of students, the greater the number of hours they must work.

Students who make the design	*Hours to work per student*
7	*3*
3	*x*

7/3=x/3

5. We have to take into account inverse proportionality. Therefore, x = (7x3)/3 =7.

6. Therefore, the 3 remaining students will dedicate 7 hours each to finish the decoration in time.

Figure 5. Incorrect solution assuming 21 hours as required. S12 response.

Assignment of Algebrization Levels

The assignment of algebrization levels was not very difficult. Most of the students did it correctly except for the following two cases: (a) student S10 assigned Level 0 Algebrization to a Proto-algebraic Level 1 solution (see Figure 6), and (b) two students assigned Level 1 to a Proto-algebraic Level 2 solution (see Figure 7). Six students justified their decision based on the degree of generality of the intervening objects: particular numbers, classes of numbers when resorting to fractions or multiples of numbers, and the presence of unknown; the operational ("operation equal to answer") or relational meaning (in equations) of the equal sign and the type of operations performed.

For this solution, the algebraic reasoning level is 0, since operations are performed only with particular numbers.

Figure 6. Level 0 Algebrization assigned by student S10 to a solution by unit reduction.

Figure 6 shows that student S10 does not recognize the greater generality degree involved in deducing the unit value that allows the number of hours to be obtained, x, as a function of the number of students, y, or reciprocally, according to $x \cdot y = 21 \cdot 7 = 141$.

To solve the exercise with this method, the algebraic reasoning level is 1, since it uses particular numbers to make a rule of 3 in which an unknown appears.

Figure 7. Algebrization level 1 assigned by student S10 to a solution by the rule of three.

Figure 7 shows that the same student (S10) does not consider operating with the unknown but with particular numbers (coefficients in the equation) which leads him to determine that the Algebrization Level is 1 instead of 2.

Onto-Semiotic Configurations

All students made onto-semiotic configurations corresponding to the strategies used and in general, had no difficulty in completing the sequencing of the elementary practices involved. For the evaluation of the onto-semiotic configurations elaborated by the students, the criteria presented in Table 2 were used.

Four students (the third part) made irrelevant configurations since they only showed the sequencing of practices without identifying the objects (they only referred to the non-presence of algebraic objects and processes in the arithmetic solutions). Only one student showed relevant configurations. The other students (seven) elaborated irrelevant configurations: the objects do not correspond to the referred elementary practices or are incorrect.

Table 2

Criteria for the valuation of onto-semiotic configurations identified by future teachers.

Valuation	Criteria
Relevant	- Understands the sequence of elementary practices and for each unit of these correctly refer to the objects and processes involved.
Little relevant	- Understands the sequence of elementary practices but not all the objects involved are referred to or some of them are not correct.
No relevant	- The sequence of elementary practices is not correct and/or the objects referred to in the elementary units of analysis do not appear.

Figure 8 shows an example of a little relevant onto-semiotic configuration developed by student S2, and Figure 9 a relevant configuration developed by student S7.

Sequence of elementary practices to solve the task	*Objects referred to in the practices (concepts, propositions, procedures, arguments.)*
A table is made where the children are placed in one column and the hours they take in the other.	*Concept: relationship, table.* *Proposition: the creation of a table.*

Then the 7 children are reduced to one unit and 7 multiply the 21 hours. *As a result, 1 child would take 147 hours.*	*Concept: relationship, unity.* *Proposition: the reason is inverse.* *Argument: it is true that fewer children take more hours.*
Finally, it is multiplied and divided by 3 and the solution is reached: 3 children take 49 hours.	*Concept: relationship, unity.* *Proposition: the reason is inverse.* *Argument: it is true that fewer children take more hours.*

Figure 8. Example of little relevant onto-semiotic configuration, given by S2.

The onto-semiotic configurations allow detecting conceptual or argumentative conflicts in the students' responses. One of the concepts that they most frequently include is repartition since they understand that the hours dedicated to decorating the classroom must be distributed in an equitable manner among the students.

Students conflict with the term ratio. As can be seen in Figure 8, student S2 gives as a proposition: "the ratio is inverse", possibly referring to the relationship between the magnitudes: "children decorating the room" and "number of hours needed"; it is of inverse proportionality.

In Figure 9, student S7's use of the term ratio, associated with that of proportion, refers to the product of corresponding quantities of inversely proportional magnitudes (hours and students). Likewise, the arguments used to justify the inverse proportionality relationship are only partially correct, making reference, as can be seen in Figure 9, that: "the conditions that define indirect proportionality are met: more students have fewer hours or the other way around".

In general, students do not correctly recognize procedures other than arithmetic ones. It is significant that several students attribute to the phrase: "algorithm of the equation", the meaning corresponding to the procedure they follow when finding the unknown in the equation associated to the inverse rule of three.

Sequence of elementary practices to solve the task	*Objects referred to in the practices (concepts, propositions, procedures, arguments.)*
1. An indirect proportionality correspondence is assumed between two magnitudes: "hours" and "students".	*Concepts: indirect proportionality, magnitudes* *Proposal: the relationship between the two magnitudes is one of indirect proportionality.* *Argument: the conditions defining indirect proportionality are met: more students have fewer hours or the other way around.*
2. Therefore, the ratio of hours that correspond remains constant *21 hours ×7 students=x hours ×3 students*	*Concepts: the ratio of hours; proportion, unknown* *Proposition: the ratios are the same* *Argument: hours by the number of people is constant*

3. Taking into account the rule for indirect proportionality would be as follows x=(21×7)/3=49 hours	*Procedure: Finding the unknown* *Argument: arithmetic properties*
4. That is, 3 healthy students will take 49 hours to finish the work.	*Proposition: The time is 49 hours, many more than with more people* *Argument: the sequence of practices 1) to 4)*

Figure 9. Example of relevant onto-semiotic configuration given by S7.

They also show difficulties with the object *proposition*; it can be seen in Figure 8, the student refers to "making a table" as a proposition. Likewise, with the object argument that they frequently use as the intentionality of the practice carried out. Thus, expressions such as "knowing the number of hours of the four sick students" or "adding the initial hours of the students to the extra hours to know the total hours" are frequently recognized as forms of argument.

Identification of Difficulties

Knowing the difficulties primary school pupils may have, provides teachers with criteria for designing and managing instructional tasks that emphasize certain concepts, achieving instructional processes with greater cognitive suitability.

The difficulties pointed out by the future teachers, according to the described strategies, are the following:

– *Arithmetic solution* (Algebrization level 0). In general, students state that they do not encounter difficulty in this type of strategy. However, five students who identify them point to possible arithmetic errors. For example, S3 considers as possible: "error in choosing the number that has to be divided"; S8 and S9 point out: "error in choosing the number to be multiplied"; errors associated with a conceptual difficulty related to the inverse proportionality relationship.

– *Reduction to the unit solution* (Algebrization level 1). The difficulty they point out, in this case, is in understanding that, in the case of the inverse proportionality relationship, the product of the corresponding quantities of magnitude remains constant. For example, E7 points out as a difficulty: "to understand the inverse proportionality of its operations as the multiplication of total hours by the number of students".

– *Rule of three solution* (Algebrization level 2). In this case, 10 students consider possible difficulties in working with unknown, such as: "they do not know how to work with literal symbols as unknowns" (S1), "they do not have a relational meaning of equality" (S4), or to correctly raise and solve the rule of three. For example, S7 considers it possible: "to choose badly the number that has to be multiplied or divided" and "to solve badly the unknown".

In addition, those students who solved the problem correctly, point out in a general way, as a possible difficulty, to consider the relationship of direct proportionality instead of the inverse. Some students also refer to “difficulties in understanding the statement” (S4), “not understanding mathematical processes in their totality but by means of rote learning strategies” (S7) or of affective type.

Analysis of the 6th Grade Primary School Pupils’ Answers

Assessment of the Solution and Arguments Given by Primary School Pupils

All the prospective teachers agreed that both the solutions and the arguments given by primary school pupils are correct. Three of them (the fourth part) did not justify their assessment and the others (nine) focused their attention on the procedure followed or the accuracy of the argument, comparing, in this case, the responses of both pupils. Two prospective teachers considered the explanation given by Pupils 2 to be better (“more complete”). However, five prospective teachers rated this pupil’s explanation as incomplete. Figure 10 shows an example of such a response, given by student S5.

> *Both the solution and the argumentation of pupil 1 are correct.*
>
> *Pupil 2’s solution is correct; he makes a rule of three standard (direct proportionality), although the argumentation is not the most correct because he fails to explain how he has solved that rule of three. Although it is true that it is a direct proportionality and that with double the amount of milk you will make double the amount of cakes and so on. He would need to explain why he does the equation that way.*

Figure 10. E5’s assessment of the solution and argumentation of primary school pupil.

Prospective teachers consider that pupils should explain the procedures followed and argue, based on the direct proportionality relationship, the strategy followed (Figure 11). In Figures 10 and 11, it is observed that S5 and S12, respectively, refer to Pupil 2’s argument: “with double milk you will make double cakes, with triple milk you will make triple cakes...therefore it is directly proportional”, which does not cease to be informal and incomplete when describing the relationship of direct proportionality between two magnitudes.

> *With respect to **pupil 1**, during the solution of the problem, he does not explain what he is doing, focusing only on the operations he has to do (division and multiplication). The first data (6), does not specify what it is, although the second one comments it (“24 cakes”).*
>
> *In the solution section, he explains the procedure followed correctly (he tells us about the reduction to the unit in his own way), even if it is badly expressed.*
>
> *With respect to **pupil 2**, we can observe how the problem is more complete and solved it correctly in another way. Unlike pupil 1, this pupil includes an explanation in which he verbally reflects that the problem corresponds to a direct proportionality, adding his own conclusions (double litres, double cakes...).*

Figure 11. Assessment of pupils’ solutions and argumentations by S12.

Assignment of Algebrization Levels

From an epistemic expert point of view, it is recognized that the Pupil 1's response follows the procedure of reduction to the unit: he obtains the number of cakes (6) that he can make with 1 litre of milk. He performs arithmetic operations (multiplication and division) with particular numbers (18 cakes, 6 litres of milk, etc.). In the argumentation to justify his answer, Pupil 1 recognizes, in natural and numerical language, the criterion for obtaining the number of cakes from the unit value: "I multiply it [the number of litres of milk] by the [cakes] he makes with one". On the other hand, the activity developed by Pupil 2 is considered Proto-algebraic of Level 2. The procedure followed is a rule of three representing diagrammatically the table:

Litres of milk	3	4
Cakes	18	X

where the directly proportional magnitudes, the known values (3, 4, 18) and unknown (x) are written. The literal symbol "*x*" is linked to the contextual information, the number of cakes that can be made with 4 litres of milk, and an equation of the form $Ax = B$ is solved.

Table 3 presents the frequencies of the values assigned by the future teachers to the algebrization levels of the pupils' solutions. One of them did not respond to this section. Five students get correctly the algebrization level of the activity developed by Pupil 1, and seven did so with the algebrization level of the solution proposed by Pupil 2. The students who assigned Algebrization level 0 to the answer given by Pupil 1 and justified their answer, made reference to the presence of arithmetic operations with natural numbers. For example, student S6 maintains that Pupil 1 "solves the problem through arithmetic operations", without identifying the degree of generality in Pupil 1's argumentation to describe how to obtain the number of cakes from litres of milk.

Table 3

Frequencies in the assignment of algebrization levels by prospective teachers.

	Algebrization levels/ Frequencies		
	0	1	2
Solution elaborated by Pupil 1	6	5	0
Solution elaborated by Pupil 2	0	4	7

On the other hand, the four students who assigned Algebrization Level 1 to the Pupil 2 solution referred to the presence of symbolic language, without distinguishing grades from transformations with unknown. For example, student S10, when assigning Level 1, affirms: "since as we can see he carries out operations in which he includes an unknown but without excessive difficulty".

Identification of Elementary Practices

We use the criteria presented in Table 4 to assess the prospective teachers' identification of elementary practices of the answers given by primary school pupils.

Table 4

Criteria for the assessment of the identification of elementary practices by future teachers

Valuation	Criteria
Relevant	- Correctly establishes the sequence of practices and the intentionality of each unit.
Little relevant	- It is limited to describing the strategy followed in each solution.
Not relevant	- In other cases.

No prospective teacher considered the justification of pupils as elementary practice, and even those who performed a more detailed sequencing only considered operational practices and not discursive (argumentation) practices. Examples of responses to this item are presented in Figures 12 and 13.

The first pupil has followed the following steps: firstly, he has divided and secondly he has multiplied.

The second pupil has made a rule of three where there is an unanswered question.

Figure 12. Not relevant answer given by S1 when identifying elementary practices.

Figure 12 presents a non-relevant answer, while Figure 13 presents a little relevant answer. In general, the response given to this last item was little (five students) or not relevant (seven students).

Pupil 1:

If you make 18 cakes with 3 litres, we will know how many cakes you can make with 1 litre: 18: 3 = 6 cakes you can make with 1 litre of milk.

If with 1 litre you make 6 cakes, with 4 litres you make: 6 x 4 = 24 cakes.

Pupil 2:

1. He/she establishes a proportion: If with 3 litres it makes 18 cakes, with 4 litres it will make x. x being the number of cakes you make with 4 litres of milk.

3 litres	*4 litres*
18 cakes	*X*

2. The equation established is the following: 18 cakes·4litros=3litres·x

3. Solution of the equation: 72= 3x; x= 72/3= 24 cakes.

Figure 13. Little relevant answer is given by S6 when identifying elementary practices.

CONCLUSIONS

In this article, a qualitative-interpretative research has been developed, whose purpose has been to describe and interpret the results of the implementation of a didactic intervention with prospective primary school teachers. This activity was aimed to promote the development of relevant aspects of the epistemic and cognitive facets of didactic-mathematical knowledge related to the proportionality teaching. Competence in onto-semiotic analysis allows the teacher to identify possible learning conflicts and to promote activities aimed at the development of proportional reasoning from primary school onwards.

The instructions given to the prospective teachers were aimed at developing the epistemic and cognitive facets of didactic analysis. In this respect:

– In order to analyse and evaluate the common and extended knowledge of prospective teachers and the competence to solve a problem using different strategies, including those that could be carried out by primary school pupils, an inverse proportionality task was proposed. In this sense, we have observed that students show deficiencies in the knowledge *per se* of proportionality, confusing an inverse relationship with a direct one.

– The identification of mathematical objects and processes and the recognition of algebrization levels are part of the epistemic dimension of specialized knowledge. The recognition of the meanings of objects is a complex activity. However, we have found that the intervention implemented improves the prospective teachers competence for: (a) carrying out the sequencing of practices in elementary units of analysis, (b) progressing in the identification of the objects and processes involved in the practices, beyond the application of algorithms in the proportionality problems solution, and (c) recognizing in a relevant way the different algebrization levels in different solution strategies.

– Analysing the procedures and reasoning developed by primary school pupils when solving a task or difficulties with certain strategies is related to the cognitive facet of specialized knowledge (students' knowledge). For prospective teachers, it is easier to perform sequencing of practices when they are the solvers than when they analyse pupils' responses. In addition, they do not identify the object argument as part of the practices. It is considered necessary for future teachers to identify mathematical practices and objects involved in student productions, paying attention to argumentation.

Another conclusion of the formative intervention described is that the developing the intended professional knowledge and competence requires longer-term interventions.

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AUTHORS' CONTRIBUTIONS STATEMENTS

M.B and J.D.G. conceived the idea presented. M.B performed the activities, and collected the data. All authors discussed the results and, through meetings, jointly elaborated the final version of the manuscript.

DATA AVAILABILITY STATEMENT

The data supporting the results of this study will be made available by the corresponding author, M.B., upon reasonable request.

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