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Knowledge Evidenced by Prospective Mathematics Teachers when Performing a Task Involving Geometry, Teaching and the Use of Technology

Yuri Morales-López[1]

[1] Universidad Nacional de Costa Rica, Heredia, Escuela de Matemática, Heredia, Heredia, Costa Rica

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ABSTRACT

The types of knowledge that prospective mathematics teachers use when faced with tasks involving technology are interesting because this knowledge can be used as a basis for analyzing the relevance of these types of activities when teaching mathematics. This investigation is intended to identify the knowledge evidenced by these prospective teachers when they carry out an activity involving geometry, technology and pedagogy. To do so, a task was designed which was used with 65 trainee teachers, and the *Technological Pedagogical Content Knowledge* model was used to analyze the information gathered. The results show that prospective teachers make use of knowledge in all of the domains and almost all of the subdomains of the model, confirming that this type of activity triggers these types of knowledge.

Keywords: Mathematics education; ICT; TPACK; prospective teachers; geometry; pedagogy.

Conocimientos que Evidencian los Futuros Profesores cuando Realizan una Tarea que Involucra Geometría, Enseñanza y Uso de Tecnologías

RESUMEN

Los tipos de conocimientos que los futuros profesores de matemática emplean cuando se enfrentan a tareas que involucran tecnología son de interés pues pueden ser utilizados como bases para analizar la pertinencia de una actividad de esta naturaleza. El objetivo de esta investigación es identificar los conocimientos que evidencian estos profesores cuando realizan una actividad que integra geometría, tecnología y pedagogía. Para esto, se diseñó una tarea que fue aplicada a 65 profesores en formación y para su análisis se utilizó el modelo de *conocimiento tecnológico pedagógico del contenido*. Los resultados muestran que los futuros profesores logran poner en evidencia conocimientos en todos los dominios y casi todos los subdominios del modelo confirmando que el tipo de actividad induce a la activación de estos conocimientos.

Palabras claves: Educación matemática; TIC; TPACK; profesores en formación; geometría; pedagogía.

Corresponding author: Yuri Morales-López. E-mail: ymorales@una.ac.cr

INTRODUCTION

The incorporation of technology into mathematics classes is a relevant topic for investigation, principally due to the strong need to familiarize students with processes such as exploring, making assumptions, and making arguments. Activities involving the use of technological resources may allow students to both become technologically proficient and develop the necessary skills that they will use in their work and daily life.

The design of activities that effectively incorporate information and communications technologies (ICTs) is one of the core competences for prospective mathematics teachers. Among other things, the development of this competence by prospective teachers will require them to be able to identify the types of knowledge needed to integrate mathematics, mathematics teaching, and technological resources. It is imperative that prospective mathematics teachers have the necessary skills to create and evaluate tasks involving ICTs.

The *Technological Pedagogical Content Knowledge* (TPACK) model (Mishra & Koehler, 2006) is a knowledge organization system for the study of the which relationships between content, pedagogy and technology, which enables studying the way in which these relationships synergistically create comprehensive and valuable knowledge for the incorporation of technological resources into mathematics classes. This model may be useful for understanding the types of knowledge that prospective mathematics teachers possess.

This research therefore intends to find out what types of knowledge prospective mathematics teachers evidence when they carry out a task using geometry, teaching, and technologies. Specifically, a task involving these three areas was applied to 65 prospective secondary education mathematics teachers (PSEMTs). Different situations that show the use of some of the types of knowledge considered in the *Technological Pedagogical Content Knowledge* (TPACK) model are analyzed in this article.

THEORETICAL FRAMEWORK

Several general ideas on the use of technologies and on different knowledge organization systems are discussed below, together with further details of the TPACK model.

Use of Technological Resources

Technology is part of day-to- day life. Students increasingly have access to technological instruments and digital devices with microprocessors such as intelligent mobile telephones, personal computers, and tablets, but learning how to use them may take time, and the use of more advanced devices may often require support or instruction from other persons.

The dynamics of learning to use these devices and computing resources such as software may be somewhat confusing when they are formally incorporated into the classroom. The use of computers or mobile telephones in a classroom does not necessarily improve the quality of a class, or help students to become digitally proficient, or to learn how to create the content expected. In general, simply including devices such as smart whiteboards, digital projectors, tablets, and telephones in teaching does not result in a real integration of technologies into the educational process. The same is true of software: if, for example, teaching students about the use of GeoGebra software focuses exclusively on how to use its buttons, make circles, scroll screen content, fill out tables, and create graphics, the expected results of incorporating technologies into mathematics education will not be obtained.

In fact, learning only the technical details of using computing software is not sufficient to lead to the construction of real mathematical knowledge. For instance, innumerable objects can be created using *Dynamic Geometry Software* (DGS) without the user forming a clear idea of their nature. In the same way, figures, graphics, tables and symbols may be manipulated with software without actually understanding their characteristics (or their invariants). It is perhaps necessary to focus more on mathematical processes when technology is being used, and on which of these processes may most appropriately involve the use of ICTs.

Different authors have indicated that technological resources may contribute to certain processes involved in teaching mathematics in the classroom. Soldano, Luz, Arzarello and Yerushalmy (2018) point out that in the specific case of geometry, DGS may provide support for investigation, exploration, generalization, verification and refutation, but these processes are inherent in mathematical activities themselves, and not in the software. This conclusion may be extended to the teaching of all types of mathematics and the software used in this teaching. These authors cite many other investigations supporting this point of view, and the topic has also been addressed in studies by Morales (2010, 2011, 2014), Morales-López (2017), and Morales-López and Font (2017, 2019).

Other topics that may be associated with the use of ICTs include the concepts and beliefs of PSEMTs concerning the role of technologies in an educational context. For instance, what type of incorporation of technology is to be expected from a teacher that is convinced that the use of technology does not help to understand mathematics? These beliefs and their impacts are currently the subjects of relevant investigations in education.

In the case of mathematics teaching, as stated by Goos (2014), much of the research on the impact of technology has been based on "*how students learn mathematics with technology, [while] less attention has been given to teachers' technology-mediated classroom practices and the role of the teacher in technology integration*" (p.140). This investigation focuses on the types of knowledge of mathematics teachers, who are responsible for determining the best methods for achieving the goals of the educational process, and must make consistent decisions throughout this process.

In terms of planning, the incorporation of technology cannot be studied separately from the process of mathematical education. No one can propose effective criteria for the appropriate or inappropriate use of technology without considering what is expected from the curriculum at a broader scope, and how the use of resources is integrated with other essential factors for activities in mathematics classes.

Even though this investigation addresses the knowledge of prospective mathematics teachers and technological resources, the rationale of this study is that such types of knowledge are useful only insofar as they are incorporated into the teaching process while maintaining an ideal balance between the most relevant factors in that process.

Some Knowledge Organization Systems

There are multiple models that seek to explain the different types of knowledge associated with teachers. Shulman (1986, 1987) states that there is a *Pedagogical Content Knowledge* (PCK) beyond the *Content Knowledge* (CK) that all teachers must have. This is important because:

> […] it identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. (Shulman, 1987, p.8)

Shulman's works have stimulated the formulation of several knowledge representation systems on different subject matters, or provided support for the development of studies of PCK. Among the most important are:

Knowledge organization systems associated with the types of knowledge of mathematics teachers

- Mathematical Knowledge for Teaching (MKT) defined and developed by Ball, Thames and Phelps (2008), and Hill, Ball and Schilling (2008) and later works. This model sets apart two domains of knowledge: *Subject Matter Knowledge* (SMK) and *Pedagogical Content Knowledge* (PCK).
- The mathematics teacher's specialized knowledge (MTSK) developed by Carrillo, Climent, Contreras and Muñoz-Catalán (2013). In this case the same domains as in the MTK (SMK and PCK) are used, although the authors rename SMK as *Mathematical Knowledge (*MK) and distinguish six subdomains associated with the sphere of mathematics education. They also incorporate beliefs as a category which cross-cuts these six subdomains: beliefs about mathematics from the MK study, and beliefs about mathematics teaching and learning from the PCK study.

- Types of mathematics teachers' knowledge derived from the *Didactic-Mathematical Knowledge and Competences* (DMKC) model (Godino, Batanero, Font and Giacomone, 2016; Godino, Giacomone, Batanero and Font, 2017) based on *The Onto-Semiotic Approach* (OSA) (Godino, Batanero and Font, 2007). Although this model does not arise directly from Shulman's work (1986, 1987), it takes advantage of the findings from PCK investigations, as well as from other models and theories (Pino-Fan, Assis and Castro, 2015).

  The DMKC model is based on three dimensions: *mathematics*, which refers to common and extended mathematical knowledge; *didactics*. which is developed based on the concept of *Didactic-Mathematical Knowledge* (DMK) (Godino, 2009), and six sub-categories of teachers' knowledge (epistemic, cognitive, affective, interactional, mediational and ecological); and lastly, a *metadidactic* dimension which is based on knowledge arising from reflecting about the practice of teaching (Font, Breda, Giacomone and Godino, 2018).

Knowledge organization systems directly associated with knowledge about technology integration

The *Technological Pedagogical Content Knowledge* model was originally known as TPCK (Mishra & Koehler, 2006) and was later renamed as TPACK (Koehler & Mishra, 2009). The model is based on three main elements – content, pedagogy and technology – and the systems arise from the interactions between these elements. The general domains are based on pairwise interactions between these elements: *Pedagogical Content Knowledge* (PCK, which is consistent with that established by Shulman), *Technological Content Knowledge* (TCK) and *Technological Pedagogical Knowledge* (TPK), while TPACK refers to a consideration of the interactions between all three elements (Mishra & Koehler, 2006). The present investigation focuses on this system.

Knowledge organization systems directly associated with mathematics teachers' knowledge and their links with technology

- The *Specialized Technological and Mathematical Pedagogical Knowledge* (STAMPK) model of Getenet (2017) arises from the MTK model and relates it to the TPACK model, whose main domains are *Technology Knowledge* (TK), *Subject Matter Knowledge* (SMK) and *Specialised Pedagogical Knowledge* (SPK) which is defined as specialized knowledge associated with PCK, although Ball et al.'s model (2008) does not define it in this way.
- Use of the TPACK model in mathematics education has given rise to the acronym TPMK, in which the general concept of "content" is replaced with "mathematics". Works such as those of Lim, Ang and Koh (2016), Ozgun-Koca, Meagher and Edwards (2010) and Koh (2018) already make use of this term.

TPACK Domains and Subdomains

It is evident from the previous discussion that there are many forms of interpreting and organizing types of knowledge. Since this investigation focuses on determining the types of knowledge shown by mathematics teachers when they interact with technology, the TPACK domains and subdomains are described in more detail below (Mishra & Koehler, 2006).

1. *Content Knowledge (CK)* (domain). This refers to the subject and concepts that mathematics teachers must teach, which are specified in course curricula. Mathematics teachers must therefore know the multiple representations of an object or type of knowledge, as well as the rationale associated with the definitions and theory that support the subject matters they must teach.

2. *Pedagogical Knowledge (PK)* (domain). This is based on the study and understanding of how persons learn, and different theories of learning. This calls for extensive knowledge about what education, pedagogy and general didactics are.

3. *Pedagogical Content Knowledge (PCK)* model (sub-domain). This type of knowledge is broadly defined in studies by Shulman (1986, 1987), and refers to the pedagogy associated with the teaching of a specific type of content. This refers to the need to redefine the discourse for each type of content to make it easier to teach. It is also important to study the way in which different types of content are organized and the different learning strategies that their teaching involves.

4. *Technology Knowledge (TK)* (domain). This refers to knowledge about technologies and their functions in common and specialized contexts. This knowledge is part of the context of teaching, and is associated with different human activities, so that this type of knowledge includes not only knowledge of objects (tools or instruments such as books, boards, notebooks, pencils, computers, and specific software programs), but also knowledge of their different uses. In the case of the use of digital technologies, this would include knowledge associated with the use of basic electronic devices and software to create documents, spreadsheets, to participation in online forums, and technical use of virtual learning, although perhaps not in a specialized manner.

5. *Technological Content Knowledge (TCK)* (subdomain). This refers to the relationship between the technology used and the content studied. It is based on the different representations of objects made possible by the technological resources used.
For instance, in the case of mathematics, it refers to the knowledge about the capacity of technology to make representations of mathematical objects, either through graphics, sketches or drawings, constructions, tables, or symbolic representations, and to the knowledge of the mathematical concepts that are involved in those constructions and representations, such as definitions, preservation of invariant aspects of objects, or types of systems (such as euclidean or non-euclidean geometry) within a technological context.

6. *Technological Pedagogical Knowledge (TPK)* (subdomain). This knowledge refers to the use of technological resources used for educational purposes. For instance,

it not only involves knowing how to interact in a Wiki or forum, but also the theory or theories that support the type of learning that takes place there. In addition to these types of theories, certain devices may have different generic uses and degrees of usefulness in particular educational processes.

7. *Technological Pedagogical Content Knowledge (TPACK)* (subdomain). This subdomain has to do with teachers' core knowledge that is necessary for them to interpret content and incorporate technology into educational processes. All the domains and subdomains are associated with this type of knowledge. The relevance of incorporating technology may be partly based on the study of relationships between the three principal domains and subdomains. Therefore, the use of technology makes sense as long as it is linked with with the study of both content and pedagogy.

METHODOLOGY[1]

This investigation was carried out using a qualitative framework based on a descriptive approach, and intends to identify, specifically, the types of knowledge evidenced by prospective teachers when they carry out an activity involving geometry, teaching, and technology.

Participants, Place and Time

Sixty-five (65) prospective secondary education mathematics teachers (PSEMT) participated in this study. They included 30 women from the Computing Resources course (MAC 404) of the program for Bachelor's and Licenciatura degrees in Mathematics Teaching of the Universidad Nacional de Costa Rica (BLEM-2017); the course is delivered during the third semester of the program of study, and is included in the area called Pedagogy, Specific Didactics and Technology.

The 65 PSEMTs were divided into three groups with different schedules, and the activity was carried out in the same campus of the University during the first semester of 2018. The activity took place between the last week of February and the first week of March; the students had therefore already had their first two weeks of technical training on the use of DGS.

[1] Permission was not requested from the CEP / CONEP System, since the investigation was carried out in Costa Rica, where regulations do not require such permission. Informed consent was obtained. The investigation is exempt from any of its consequences, including full assistance and possible compensation for any damage to any of the participants in the investigation as a result of participation, in accordance with Resolution No. 510 of April 7, 2016, of the Consejo Nacional de Salud.

Instrument

An activity was designed in which the presence of/use of certain types of PSEMT knowledge could be detected. The task was associated with the construction of a geometric figure using GeoGebra software, in which they were required to follow a series of instructions to explain the correct way to construct a figure to secondary education students. Table 1 shows the sequence designed.

Table 1
Instructions for the activity.

Activity
1. Watch the following video in YouTube about the construction of a rhombus https://www.youtube.com/watch?v=v0oaBu3VnA4
a) Write a possible definition of a rhombus based on that construction
b) Construct a rhombus using GeoGebra, whose diagonals measure 6 cm and 10 cm (do not use the perpendicular bisector or midpoint buttons)
2. Watch the following video in YouTube https://www.youtube.com/watch?v=qnR2IbM7bJU
a) Construct a rhombus with sides 3 cm long and a 4 cm diagonal using GeoGebra, based on the construction shown in the video.
b) Construct a rhombus with sides 8 cm long and a 6 cm diagonal using GeoGebra and based on the construction shown in the video.
c) Construct a rhombus with sides 2 cm long and a 10 cm diagonal using GeoGebra and based on the construction shown in the video.
3. Watch the following video https://www.youtube.com/watch?v=X9rRuGYdzmo
a) Write a possible definition of a trapezoid based on the construction you watched
b) Based on the video and your reply to point 3.a, provide instructions to help a student to construct a trapezoid using the GeoGebra software.
4. Share the instructions in the Moodle forum.
5. Once you are in the forum:
a) The teacher will assign you the instructions of another classmate.
b) Use the instructions proposed by your classmate to construct a rhombus using GeoGebra.
c) Explain the relevance of the instructions. You must be clear and may use different resources to analyze the instructions and offer comments to improve them.
d) Create a second version of the instructions, based on the recommendations of your classmate, and justify the changes.

Protocol

The PSEMTs were required to work in the computing laboratory individually. During the first phase they were asked to watch a video about the construction of a rhombus using a compass and a ruler. They were then asked to write a definition of the geometric object based on that construction, and later on, they were required to construct a rhombus with predefined measurements using GeoGebra.

During the second phase of the activitity the PSEMTs were asked to watch a video in which the construction of a rhombus based on diagonal measurements was explained.

They were then asked to carry out four exercises (2a, 2b, 2c, and 2d) in which they constructed a rhombus with certain measurements.

In the third phase they watched a video about the construction of a trapezoid using a ruler and compass. Likewise, they were asked to try to write a definition of a trapezoid based on what they watched. The inputs for the analyses carried out in this investigation were generated in this phase. In this case, PSEMTs were asked to write instructions for a secondary education student to be able to construct a trapezoid using GeoGebra (**first version of the instructions for constructing a trapezoid**).

During the fourth phase of the activity, PSEMTs posted their instructions in a Moodle forum; the teacher then assigned another classmate to read them and to use them to construct the required figure. The second student then provided feedback to the student that originally designed the instructions so that he or she could improve them (**second version of the instructions for constructing a trapezoid**). This phase took place a week later, asynchronously, using Moodle.

DATA ANALYSIS

The types of knowledge exhibited by PSEMTs when they carried out this activity are described in general terms below. As mentioned in the discussion of the theoretical framework, TPACK's domains and subdomains were used and the analyses are based on the trapezoid construction activity. It should be noted that each of these types of knowledge considered individually suggests extensive research agendas. Therefore, a general review of the interesting aspects of each of them is presented, without pretending to be exhaustive.

1. *Content Knowledge (CK)* (domain).

One of the elements that frequently arose from the analyses by PSEMTs of their classmates' instructions for constructing a figure is that they use the concept of measurement to locate points. For instance, it is mentioned that the midpoint of a segment must be located (Table 2).

Table 2
Comparison of different instructions written by a PSEMT.

Instructions in the first version	Suggestion or comment of the other PSEMT
We measure to the midpoint of that segment and we call it V.	Construct a midpoint of the segment TU. (There is not an exact measurement of the segment, because the idea was to not use measurements)

This also clearly occurs with aspects of the euclidean geometry that they are learning, and the language used in the software. For instance, the words *construct, create,* and *place* were used repeatedly by participants without questioning the relationship between the mathematics they know and the mathematics the software allows them to carry out.

Likewise, some mathematical language is also used in the suggestions (Table 3).

Table 3

Suggestions by a PSEMT.

● Use the correct terms; i.e., instead of using words such as *place*, *join*, it is better to use mathematical terms such as *draw a segment*, *construct*.
● I recommend that you avoid using words such as *left* or *right* when you are referring to geometric constructions.

Another point that is not addressed by PSEMTs, and which they do not understand, is the concept of parallelism. For instance, in the instructions they were asked to create a line parallel to a segment without indicating that this parallel line had to go through a point that had been defined before. Likewise, there were multiple errors when instructions were given about how to create a circle given a line segment, since it was not mentioned whether the segment was the radius or the diameter of the circle.

2. *Pedagogical knowledge (PK)* (domain)

In the case of this type of knowledge, all students tried to write instructions in the same way that they did during phase one: through continuous ordered steps in a hierarchical fashion, using a combination of symbolic representations and common language.

None of the students chose to write their instructions based on the use of other visual aids or different presentations, showing a marked instructional or sequential tendency.

3. *Pedagogical Content Knowledge (PCK)* (subdomain)

Results of carrying out the activity showed difficulty in ordering ideas to make them easier for students to understand. Furthermore, as mentioned previously, all PSEMTs assumed that writing a sequence of instructions was the best way to describe how to construct a specified figure. Table 4 presents the opinion of one of the PSEMTs with respect to the difficulty of understanding the way in which ideas are expressed.

Table 4

Suggestion by a PSEMT.

Simply carrying out step 1 was very difficult, mostly because the language used does not permit following the instructions for constructing the trapezoid. It is necessary to analyze every step in depth; maybe the instructions were intended for persons that understand mathematical language, but at a didactic level and for secondary education students that are just getting acquainted with GeoGebra, this series of steps may be quite frustrating.

In this case, the PSEMT that wrote the instructions mentions to their classmate in the Moodle forum that he has found a simpler way which he describes as *shorter and less complex* to be able to construct the figure, and shares the GeoGebra construction scheme to demonstrate the simplicity of the construction (Table 5).

Table 5

Construction protocol proposed by a PSEMT as a response to the suggestions of his classmate.

N.º	***Name***	***Description***
1	*Point A*	
2	*Point B*	
3	*Segment f*	*Segment [A, B]*
4	*Point C*	*Point on f*
5	*Point D*	*Point on f*
6	*Straight line g*	*Straight line through perpendicular C to f*
7	*Straight line h*	*Straight line through perpendicular D to f*
8	*Point E*	
9	*Straight line i*	*Straight line parallel to f through E*
10	*Point F*	*Intersection of i, g*
11	*Point G*	*Intersection of h, i*
12	*Segment j*	*Segment [A, F]*
13	*Segment k*	*Segment [F, G]*
14	*Segment l*	*Segment [G, B]*

4. *Technology Knowledge (TK)* (domain)

A general diagnostic instrument was applied to two groups during the first week of the beginning of the course to obtain basic information about prior knowledge of computers and their use. The indicators were: the computer and its parts, use of internet, and knowledge about some software programs.

At least 60% of the PSEMTs use a computer more than twice a week. With respect to the parts of the computer, more than 97 % of them easily recognize a monitor, keyboard, mouse, CPU, USB ports and the power switch. Less than one third of them know what a hard disk or RAM memory is. As for use, 98 % indicate that their main use of the computer is for browsing the internet (checking e-mails, visiting social networks, searching for information, watching videos, etc.). With respect to software programs, more than 95 % indicate that they are acquainted with Microsoft Excel and 40 % have seen somebody else use GeoGebra or knows about it.

5. *Technological Content Knowledge (TCK)* (subdomain)

The results of analyzing the instructions analyzed show that some PSEMTs use adequate mathematical symbols, while others used an inappropriate combination of symbols and software language, resulting in inconsistent explanations.

This is a significant finding, given that some PSEMTs were able to create a new representation of mathematical objects constructed in the language of the corresponding software, and were able to move between presentations (conversions between systems) at their convenience, and even use them as justification for arguments (Figure 1).

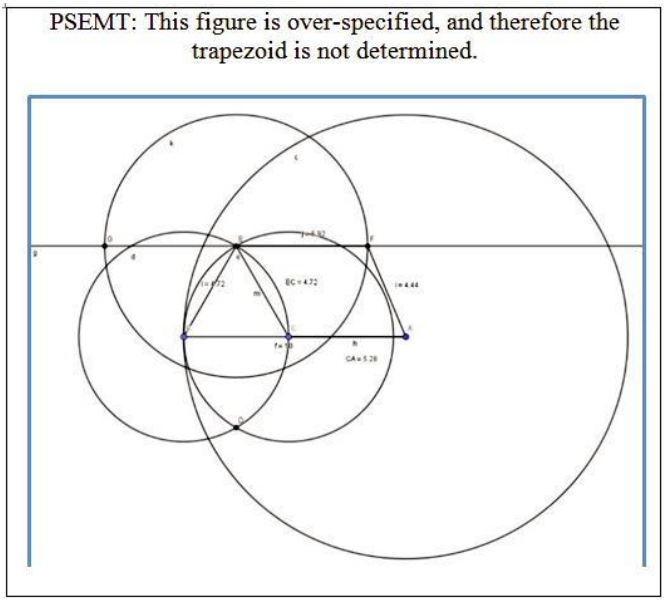

Figure 1. Graphic justification by a PSEMT about errors in the instructions of a classmate.

As shown in the discussions of previous categories, the same PSEMT started to communicate intuitively with different representations, mainly using the graphic tools and construction protocol generated by the software (as shown in Table 5).

In some cases, when PSEMTs analyzed the instructions, they determined that they did not match the capabilities of the software step by step. For instance, in the instructions they were asked to create a segment that was parallel to another, but this could only be achieved through the construction of at least one segment and the construction of another straight line passing through an external point, to then define a segment.

6. *Technological Pedagogical Knowledge (TPK)* (subdomain)

With respect to learning related to the use of this type of technological resources, it is not possible to infer whether PSEMTs are assuming or adopting a learning theory that ensures that students will properly understand the content they worked on. In spite of this, PSEMTs were aware of the need to anticipate errors and the range of possible interpretations that students could make when reading instructions.

The feedback provided PSEMTs with suggestions associated with learning how to use software in a mathematical activity. For instance, assuming that secondary education students do not have any other information available, it was recommended to use the images on the buttons and even define them and label with them with names, to be able to refer to that label when the object is mentioned.

7. *Technological Pedagogical Content Knowledge (TPACK)* (subdomain)

As mentioned in the discussion of the theoretical framework of this investigation, this is the type of knowledge that is needed for the adequate incorporation of technologies into the teaching of content.

Some PSEMTs included fundamental elements in their feedback. The need to modify instructions was not only considered from a technical point of view, but also from the perspective of the specification of a mathematical object in a way that will allow its construction to be understood.

In this sense, the different feedback comments included an adequate combination of suggestions that involve three perspectives: a) the way to construct a trapezoid using GeoGebra, b) the correct construction of a trapezoid, and c) ways to provide support for students so that they can correctly construct a figure while understanding what they are constructing.

DISCUSSION

This section presents three perspectives.

1) ***A PSEMT reformulated his instructions in terms of making the language easier to understand***. Originally, a PSEMT used mathematical language at a university level for secondary education students which, at least at the pre-university level of Costa Rican education, is not appropriate. For example, although the PSEMT was correct when he used expressions such as: "*Use the compass tool and draw a circle C(A,d(E,D))*", based on the feedback received, he reformulated his instructions to use less symbolic language. The PSEMT never doubted the importance of knowing and using terms, notations and symbols, but the most notable finding was that he decided to reformulate the instructions in simpler terms.

2) ***A PSEMT reformulated the steps used in the software so that they would be simpler than those shown in the original video.*** In this case, he avoided the use of circles to leave only the concepts of parallelism and perpendicularity, resulting in a smaller number of software instructions and tools used (even if only one step was eliminated) (Table 6).

Table 6

Comparison between the original instructions proposed by a PSEMT and their reformulation after receiving feedback.

Original wording	Reformulation of the instructions after receiving feedback
Considerations: Segments AB, CD, ED, FG $d(F,G)<d(E,D)<d(C,D)<d(A,B)$ C(r,O): Circumference with center O and radius r	1) Construct the segment AB. 2) With the *Point* tool, construct points C and D on the segment AB. (They should preferably be aligned in the following order A, C, D, B)
Steps to follow: 1) Draw a segment with the same measurement as the longer side. Call it AB	3) Construct the straight line g, so that it is perpendicular to AB and contains point C. 4) Construct the straight line h, so that it is perpendicular to AB and contains point D.
2) Use the compass tool to draw a circle C(A,d(E,D)). 3) Use the Intersect button and look for the point at which the circle created in the previous step intercepts the segment AB. Call it X	5) Construct the straight line i, so that it is parallel to AB. (This straight line i will generate a point E; place it anywhere except between the straight lines g and h).
4) Just as in step 2), use the compass tool and construct C(B,d(F,G)) 5) Construct another circle using the compass tool so that C(X,d(F,G))	6) Using the Intersect tool, mark the intersection point F of the straight line i with the straight line g.
6) Use the Intersect button and indicate the point of intersection between the circles of the two previous points. Call it Y. 7) Draw the segment between point X and point Y, and, in an analogous way, between B and Y.	7) Using the Intersect tool, mark the point of intersection G of the straight line i with the straight line h. 8) Hide the straight lines g, h, i.
8) Use the parallel button and construct the parallel to AB and that contains Y. 9) Analagously to what wass was done in the previous point, draw a straight line parallel to XY that contains A. 10) Use the Intersect button to determine the intersection between the parallel lines drawn in steps 8) and 9).	9) Construct the segments AF, FG, and GB Observe that the quadrangle AFGB is a trapezoid. Note: The point E generated in step 5) determines the length of the internal height segment.

3) ***A PSEMT attempted to think of a vision other than that of a mathematical object to make the activity simpler.*** The previous point is also related to the mathematical object because, although the figure constructed has the shape of a trapezoid, an important problem arises when it is repositioned on the screen. For instance, if the point used to reposition (drag) the image is *A* or *B*, the result is an enlargement or contraction of the object. As indicated by the PSEMT in the final note he writes, this creates a dependency on an external point E which is not part of the mathematical object (Figure 2).

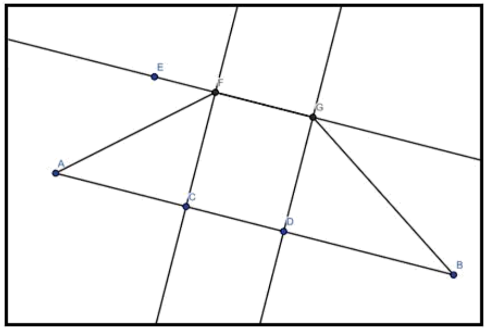

Figure 2. Image of the construction of the trapezoid proposed by the PSEMT in the second version of his instructions.

It is necessary to emphasize that looking for new definitions for mathematical concepts is not the objective of the activity nor of the TPACK model, which is rather to stimulate PSEMTs to become familiar with the integration of technology, teaching, and content, utilizing all of these types of knowledge and trying to balance their use in the activity.

FINDINGS AND FINAL CONSIDERATIONS

PSEMTs were found to use different types of knowledge when they carried out the activity, reflecting the content of the CK, PK, PCK, TK, TCK, and TPACK models. The only model for which there is no clear evidence is that of Technological Pedagogical Knowledge. Even though some PSEMTs were familiar with software, they were not prepared to effectively specify the instructions needed to teach students how to construct a geometric figure. However, many of them exhibited an adequate spatial sense, being able to describe characteristics and invariants in their feedback and offering useful suggestions.

Several elements appear in a cross-cutting manner, such as various types of language: mathematical language, language about the use of a technological resource such as GeoGebra, and the language used when explaining to others a sequence of steps so that learning takes place.

The use of certain types of knowledge were also shown when PSEMTs were asked to evaluate their peers, or were evaluated by them. Although feedback comments were not often stated formally (since they were communicated between peers), they were useful as catalysts that allowed PSEMTs to make their own corrections to instructions, and to take into account factors that they did not consider in the first version of their own instructions. Even though this was a short activity, it allowed useful discussion and collaboration.

The language, the symbology used, and ways of explaining how to work with software were the elements most frequently encountered; however, as discussed previously, there were also reflections about mathematical objects and the way to explain to a student how to construct them.

In most cases, the second version of the instructions focused on improving the language used to make it more understandable, correcting errors such as over-specification of objects, or associating the steps described more clearly with the tools and buttons of GeoGebra. This is essential in the design of sequences of activities.

Finally, it is clear that if mathematics teachers do not question themselves about the ways in which they should orient the activities involving ICTs, the language that must be used, and the mathematical concepts involved, they will not be able to properly incorporate technology in mathematics classes.

The results of this investigation may be used as the basis for defining activities that avoid the indiscriminate use of technology, and which allow the evaluation of anticipated results, as well as the consistency of the activities that are presented to prospective mathematics teachers.

Regarding the use of technology, it is clear that the degree to which its use in the teaching of mathematics is truly effective will depend on the planning of how these resources are to be incorporated. The results of this investigation illustrate the need to base this planning on a firm understanding of the types of knowledge that are required to generate activities that contribute to improving mathematics education.

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