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Learning Situation for the Trigonometric Fourier Series from a Socio-epistemological Stand Point

Rosa María Farfán Márquez[1]
Fabián W. Romero Fonseca[1]

[1] Instituto Politécnico Nacional (Cinvestav-IPN), Centro de Investigación y de Estudios Avanzados, Departamento de Matemática Educativa, Ciudad de México, Ciudad de México, México

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ABSTRACT

This article aims at showing a didactic proposal that favours the signification of the Trigonometric Fourier Series through a learning situation, based on the Socio-epistemological Theory of Mathematics Education, on research where this knowledge has been problematized. The Fourier Trigonometric Series is a complex topic for learning at a higher level, where the process is usually mechanised without fully understanding its operation and characteristics. We want to verify that with activities that support the relationship between algebra and geometry, making use of GeoGebra – dynamic geometry software – as a control variable, the series and its convergence can be signified using a physical-geometric context.

Keywords: Trigonometric Fourier Series, Convergence, Socio-epistemology, Learning Situation.

Situación de Aprendizaje para la Serie Trigonométrica de Fourier desde la Teoría Socioepistemológica

RESUMEN

El objetivo de este artículo es mostrar una propuesta didáctica que propicie la significación de la Serie Trigonométrica de Fourier a través de una situación de aprendizaje, cuyo fundamento se basa en la Teoría Socioepistemológica de la Matemática Educativa, en investigaciones donde se ha problematizado este saber. La Serie Trigonométrica de Fourier es un tema complejo para su aprendizaje en el nivel superior, donde por lo general se mecaniza el proceso sin comprender del todo su funcionamiento y características. Se quiere comprobar que con actividades que apoyen la relación entre lo algebraico y lo geométrico, haciendo uso de GeoGebra – software de geometría dinámica – como variable de control, se puede significar a la serie y su convergencia mediante un contexto físico-geométrico.

Palabras clave: Serie Trigonométrica de Fourier, Convergencia, Socioepistemología, Situación de Aprendizaje.

Corresponding author: Rosa María Farfán Márquez. E-mail: rfarfan@cinvestav.mx

INTRODUCTION

One of the fundamental purposes of Mathematics Education is the improvement of learning processes. However, there is continuing concern about bringing research results into the classroom. In view of this concern, the design of learning situations, based on research, has been an ideal tool for contributing to schooling processes.

On the other hand, mathematical knowledge is not a "static" entity; it has evolved along with human beings, affected by socio-cultural environments from its emergence until its arrival into education systems. This leads epistemological considerations to play a crucial role in the design of didactic intervention situations since, by understanding the conditions of their emergence, we will be able to make better articulations among the knowledge for the classroom (Farfán & Romero, 2016).

In the case of the Trigonometric Fourier Series (TFS), in addition, to be a fundamental subject in advanced engineering courses (Muro, Camarena & Flores, 2007), it was historically a turning point for the development of Mathematical Analysis and an essential part of the evolution of the concept of function. As a result, numerous investigations have been concerned about various aspects related to the series:[1] The problem of the vibrating string as an antecedent of the TFS (Farfán, 2012; Ulín, 1984), the determination of the stationary state as intrinsic phenomenology to the series (Farfán, 2003, 2012; Marmolejo, 2006); and the physical and mathematical notions related to the TFS such as the activity of modelling (Morales, 2013; Morales & Farfán, 2004), the notion of heat (Morales, 2010), the mathematical visualization of the TFS (Rodríguez, 2010; Rodríguez & Popoca, 2010), the hypothesis of periodicity (Vásquez, 2006) and the convergence of the TFS (Moreno, 1999)

In addition, Montiel (2011) places the TFS as the most advanced stage of trigonometric functions in the development of trigonometric thinking, where the sine and cosine functions must be constructed as objects by students, in other words, they must be susceptible to manipulation.

Thus, this document is concerned about providing a detailed presentation of a learning situation for TFS, whose initial concern was to signify the mathematical notions around the series. The foundation of this design is based on the results of three decades of socio-epistemological research. Although it is true that it is expected to contribute to the educational problem regarding the learning of the TFS, we know that a design will not solve all the problems that it entails, but it does offer an essential contribution in this respect.

SOME ELEMENTS OF SOCIO-EPISTEMOLOGY

The Socio-epistemological Theory of Mathematics Education (STME) considers mathematical knowledge as an emergent element of social dynamics, where "the starting

[1] An analysis of the state of the art with respect to the Fourier series can be found in (Romero & Farfán, 2016).

point for the construction of knowledge is the activity regulated by emergent elements of a social nature that we call *social practices*" (Cantoral, 2013, p.48). In other words, mathematical knowledge is generated socially through situated practices, where the notion of social practice is the regulator of all human activity, i.e. social practice is not what people do, it is what makes them do what they do, even when they are not aware of their own actions (Cantoral & Farfán, 2004).

Meanwhile, mathematical knowledge was socialized in non-school environments, so its introduction into teaching systems causes knowledge to change its structure and functionality (Cantoral, Moreno-Durazo & Caballero-Pérez, 2018); when knowledge reaches school, different discourses are produced that change the organization and functioning of mathematical knowledge, the STME has called them School Mathematical discourse (SMd) (Cantoral, 2013).

> Given the nature of SMd as a motionless, meaningless, utilitarian entity that favours the exclusion of the social construction of knowledge, the aim is to provide an epistemology – the socio-epistemology –, which enables the redesign of SMd from the problematization of mathematical knowledge with a view to promoting the learning of this knowledge. (Farfán & Romero, 2016, p.118)

It is essential to carry out a systemic study of SMd in order to understand the role it plays in the teaching system, so its redesign can be proposed considering that it is the learners who must build their own knowledge by making knowledge work, in other words, knowledge is a means for decision making when solving the problem posed in a *learning situation*. However, how can we ensure that the right environment is created for this?

According to Cantoral (2013), you are not always in a learning situation. Cognitive conflict is a way of fostering it, proposing a problem situation that confronts the subject with a scenario in which she/he must deploy the knowledge that is required from a cognitive disequilibrium (Piaget, 2009). However, this position is shared by different theoretical approaches in our discipline, so what characterises a socio-epistemological learning situation? According to Reyes-Gasperini (2016), the learning situation should favour a pragmatic evolution of mathematical knowledge (school mathematical know-how), bringing mathematical know-how into play through the context of significance (Figure 1).

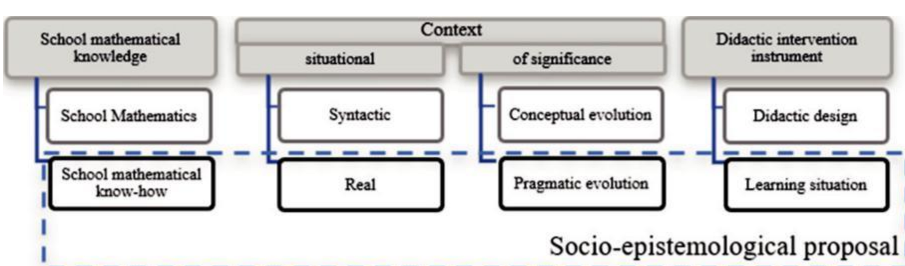


Figure 1. Learning situation (Reyes-Gasperini, 2016, p.61).

A distinction is made between the situational context and the context of significance. The first refers to the way of contextualizing the task, which can be through a known environment or medium, but the mathematical structure is not affected by dispensing with it or altering it – syntactic context –, another way of contextualizing is with a scenario intrinsic to the task that raises a situation that requires mathematical knowledge to respond, giving mathematical knowledge meaning through its use – real context –. Meanwhile, the context of significance is the way to contextualise the construction of mathematical knowledge, which can be through algorithms or as a consequence of a change of representation – conceptual evolution –, among others; or as a result of practices and signification through use – pragmatic evolution – (Reyes-Gasperini, 2016).

This pragmatic evolution precedes and accompanies the conceptual evolution of mathematical knowledge – school mathematics –. Therefore, the learning situation becomes a didactic tool for the social construction of mathematical knowledge, as it promotes the signification of mathematical objects through use.

DIDACTIC ENGINEERING

Didactic Engineering (DE) functions as a guide for the design of situations for application in the classroom, as well as a research methodology that guides classroom experiments (Farfán, 2012; Artigue, 2014, 2015). The DE emerges with the aim of improving the understanding and functioning of the didactic systems, paying particular attention to the limitations and variables that act on the system and for this purpose, special attention is paid to controlled didactic realizations, which have a prominent role in the methodology of the DE to validate the created situation designs (Artigue, 2014).

The DE has four phases, which correspond to its experimental working scheme: preliminary analysis; situation design and a priori analysis; staging, observation and data collection; a posteriori analysis and internal validation. This paper pays special attention to the second moment: the design of the situation and a priori analysis.

Preliminary Analysis

Researches framed in the STME analyse the role of social practice in the constitution of knowledge systemically, taking into consideration four fundamental components regarding knowledge: its epistemological nature, its sociocultural dimension (since knowledge is a social and cultural construction), the planes of the cognitive and the modes of transmission via teaching (Cantoral & Farfán, 2003).[2]

The integration between these four components is what STME calls a *problematization of mathematical knowledge*, which lies in “making knowledge a problem

[2] It is important to emphasize that the DE, from its origins in the French school of didactics of mathematics, considers only three dimensions of knowledge: epistemological, cognitive and didactic. It is in the STME where the socio-cultural dimension is added, which causes great changes in the study of the other dimensions.

through its four dimensions, a subject for analysis, locating and analysing its use and *raison d'être*, i.e. studying the nature of knowledge" (Reyes-Gasperini, 2016, p.52). Thanks to the problematization of knowledge, DE methodology is strengthened.

For this research, this analysis takes into account the results obtained by a group of investigations into the notions surrounding the TFS, its historical-cultural context of origin, the mechanisms of institutionalization via teaching and the cognitive processes associated with tasks in which this knowledge is put into operation, a detailed analysis can be found in (Romero, 2016; Farfán & Romero, 2017).

Situation Design and a Priori Analysis

Learning situations are designed based on the preliminary analysis, and for this purpose, Didactic Engineering supposes the selection of diverse aspects, acting on different variables of the system, that are appropriate for the problem posed. The types of variables include:

- Macro-didactic or global: they refer to the global organisation of engineering. It is related to decisions, such as resorting to computer tools, prior knowledge, the predominance of one frame of reference over another (numerical, graphical, algebraic, analytical), education system, institutional policies, curriculum, among others.
- Micro-didactic or local: concerning the organisation of the learning situation, i.e. the description of the process to be followed (specific problems, size of the working groups, discussion times, among others). Furthermore, within the situation, transits must be carried out in different contexts: numerical, algebraic and geometric, in order to thus detect the conditions that permit more optimal operations.

It is essential to point out that, although the selection of global variables is usually presented separately from local variables, they are not independent of each other, since general conceptions must allow the evolution of local ones, which are directly linked to the design of the learning situation.

Therefore, the a priori analysis includes a description of the local selections relating them to the global ones and the characteristics of the learning situation; afterwards, it is analysed what could be at stake during the development of the tasks: possibilities of action, of selection, of decision, of control and of validation available to the student; therefore, it is sought to predict that the expected behaviours, if they occur, are the result of putting into practice the knowledge contemplated for learning and that the task was trying to develop.

Subsequently, experimentation, observation and data collection are carried out and, based on these, the a posteriori analysis, for the purpose of validating the intervention design. However, as mentioned earlier, this article only reports on the design of the situation and the *a priori* analysis.

THE LEARNING SITUATION

Based on the preliminary analysis, that can be found in (Romero, 2016; Farfán and Romero, 2017), macro-didactic variables are established, which will guide the general behaviour of the learning situation, that include:

- Functional character: The TFS should be presented as a *prediction* tool when *modelling* and *interpreting* certain phenomena close to the subject (individual or collective), recognising the *Prædiciere*[3] as the social practice that governs its construction.
- Diverse contextual rationality: This takes into consideration the evolution of trigonometric thinking, from the function to the series, which will allow the emergence of arguments in the context of the learner, since the trigonometric function must be resignified in order of the trigonometric series to emerge from the study of convergence.
- Validation of knowledge: Behind the TFS there is a diversity of arguments (physical, geometric, analytical and algebraic), so this diversity must be taken into consideration when this knowledge is constructed.
- Plurality of reference practices for resignification: The construction of the TFS requires the *modelling* and *interpretation* of a stationary phenomenon of periodic and bounded variation, for which the TFS becomes a *prediction* tool. On this basis, its interaction with different contexts must be identified.

Meanwhile, the target population for whom the situation was designed are students of the bachelor's degree in Physics and Mathematics of the National Polytechnic Institute in Mexico. Since the phenomenological environment in which the Fourier series emerged – the propagation of heat – is cognitively more complex than the series itself (Farfán, 2012), it is not a suitable environment as a context of significance for the series. A context of significance close to the target population needs to be sought, for which the modelling of a physical-geometric type phenomenon is proposed: the superposition of circular movements.

To date, this design has been piloted repeatedly: the first time with a group of students from the master's program in Mathematics Education of the Cinvestav-IPN in order to review writing, clarity of ideas, execution time, among other factors; the second time it was implemented in a workshop with teachers from the upper middle and higher level, seeking to check whether the corrections made based on the first piloting were successful. A more recent piloting was carried out with three graduates from the bachelor's degree in Physics and Mathematics of the National Polytechnic Institute in Mexico, in order to obtain a point of reference of the target population. Although the objective of a pilot is not to analyse the productions of the participants – the a posteriori analysis phase of the

[3] A detailed analysis of Prædiciere as a social practice can be found in (Cantoral, 2013).

DE is used for this –, references are made to them in the following sections in order to validate the relevance of the proposal in some way.

The situation is organized based on the two moments of social construction of the TFS proposed by Romero (2016): 1. understanding that a trigonometric series can converge to a function (Tasks 1 to 5) and 2. given a function that can be represented in a trigonometric series, determining the coefficients of the series (Task 6). Task 1, 2, 4 and 6 are presented below, detailing the intent of each subsection.[4]

Introduction: The Movement of the Planets

This part is intended to familiarise the student with the model of planetary motion proposed by Alexandrian astronomers.[5] This consisted of a circumference centred on the Earth and a point that moves on its perimeter. This point is the centre of another circumference, and another point moves on the perimeter of that circumference, which is the centre of another circumference and so on. All points move counter clockwise with uniform angular velocity. This model of movement is known as the superposition of circular movements or epicycles (Figure 2).[6]

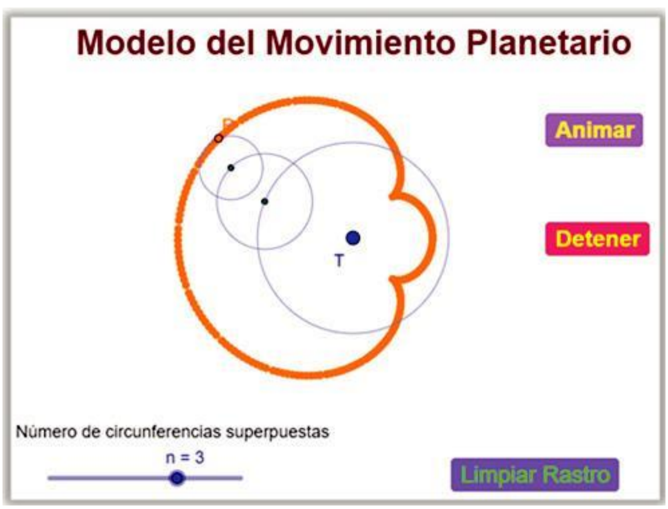


Figure 2. Applet of introduction to the learning situation (https://ggbm.at/yGBB3pVF).

Here, and for the entire design, the GeoGebra applets used work as a control variable to promote a deeper understanding of the phenomenon. This is necessary from

[4] The complete situation – spanish version – can be found in GeoGebra's book: https://ggbm.at/byc8hxdv.
[5] It should be noted that at that time the Sun and the Moon were considered planets so, during all tasks when reference is made to planets, this also includes the Sun and the Moon.
[6] For reference of this model and to extend the introduction, it can be revised (Calles, Yépez & Peralta, 2003).

the preliminary analysis (Farfán & Romero, 2017) since the first sensitive experience is not sufficient to model steady-state phenomena, which requires a deeper understanding of them (Farfán, 2012).

Some Tasks of the learning situation are presented below in order to analyse the type of questions proposed, as well as the route to follow, starting from an in-depth qualitative understanding of the phenomenon towards its mathematization, where coherence between the physical phenomenon and the mathematics involved need to be observed, just like Fourier did in the case of the heat propagation problem.

Task #1: Explaining the Movement of the Planets

The objective of this task is to *characterise the behaviour of the system in a qualitative way, which will allow a deeper understanding of the phenomenon and not only what the senses detect with the naked eye*. This Task is divided into two parts; the first (Figure 3) tries to understand why the Alexandrian model allows us to explain what the Greek model did not allow. The aim is to provide a deep understanding of the system, just like Fourier knew the behaviour of the heat propagation phenomenon through an empirical approach before proposing his mathematical model of the phenomenon (Farfán & Romero, 2017).

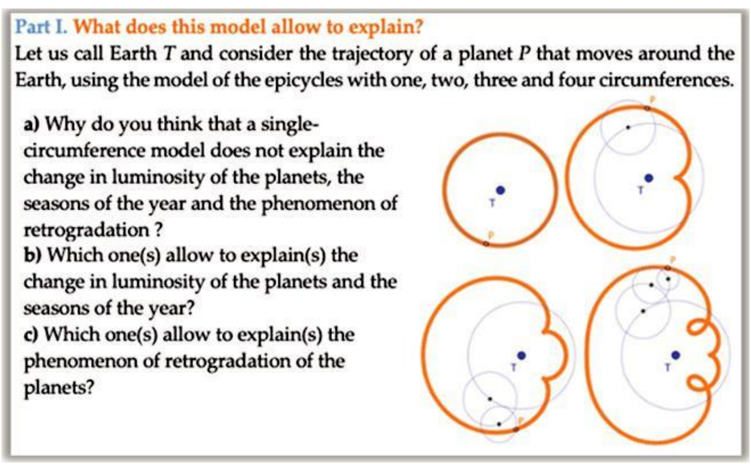


Figure 3. Task #1 – Part I, items *a*, *b* and *c* of the learning situation.[7]

7 The change in luminosity refers to the change in brightness of the planets. In the time of the Alexandrians it was believed that this was due to the change in the distance between the Earth and the planet. This same explanation was given for the seasons of the year, remember that at that time the Sun was considered a planet. The phenomenon of retrogradation refers to a brief interruption in the movement of some planets during brief intervals, where movement occurs in the opposite direction to the usual movement.

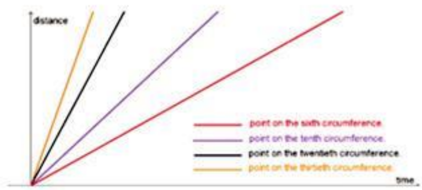
The student is expected to relate the change in luminosity and the seasons of the year to the distance between *T* and *P*; therefore, the model of a single circumference does not permit the explanation of these phenomena since this distance never changes. Regarding the phenomenon of retrogradation, its explanation shall be in the loops that are generated in the model with 4 circumferences.

Based on the piloting carried out and several uncontrolled staging, we noticed that these questions confront the student since they are forced not only to say what is happening, but also to explain the causes of these behaviours, which provoked a cognitive demand, beyond what perception allowed them to observe in the images initially.

The second part of this task seeks, with the support of an applet (https://ggbm.at/ d3CN5DT8), to glimpse the notion of stability of the system. For this purpose, the applet allows to increase the number of circumferences to up to thirty and thus to analyse the changes that occur in the behaviour of the system.

Part II. What if we increase the number of epicycles?
Let us call Earth *T* and consider the trajectory of a planet *Q* that moves according to the model of epicycles as shown in the provided applet.

a) How do the radii change from one circumference to another as they are added?
b) According to the graph, how does the movement of these points change from one circumference to another?



c) Using the applet provided, explain in your own words how does the trajectory of the planet change as more and more circumferences are added?

d) Do you think there is a relationship between your answer to question (c) and what you answered in questions (a) and (b) in Part II? Why yes? or Why not?

Figure 4. Task #1 – Part II, items a to d of the learning situation.

Item *a* (Figure 4) seeks to conclude that the radii of the circumferences tend to zero in a qualitative way, and for this reason, students could use phrases such as “decrease”, “become smaller”, among others. In item *b* (Figure 4), the graph provided shows the distance travelled based on the time for each point on the sixth, tenth, twentieth and thirtieth circumferences. By means of the study of change, students are expected to infer that the speed of the points on each circumference continues to increase without bound and, for this purpose, they could use phrases such as “goes faster”, “in the same time it travels more and more distance”, among others.

According to the pilot test, it is possible that for question *b*, the students consider that the graph refers to the trajectory of the planet when adding circumferences, when in fact it refers to the movement of each point on each circumference and this clarification should be made.

For item *c* (Figure 4) the intention is that the student identifies the stable nature of the system in a qualitative way, using phrases such as “when there are many circumferences, the shape of the trajectory hardly changes as more are added”, “tends to resemble a...”. For this purpose, the change in the shape of the trajectory is expected to be analysed as more and more circumferences are added.

During piloting and staging, the participants were able to identify the stable nature of the phenomenon. However, they stated that “something strange happened on the left part of the trajectory”, which is relevant because not all of the trajectory is stable, and this will give meaning in subsequent tasks to the notions of convergence and divergence of series.

Item *d* (Figure 4) seeks to compare how the trajectory of the planet changes (stability) with what causes the change. However, determining how it changes and what produces the change is a cognitively very complex task for steady-state determination phenomena (Farfán, 2012; Romero, 2016). It is hoped that, with the help of the teacher and based on the ideas proposed by the students, they will come closer to the idea that by making the radii of the circumferences tend to zero and the speed of movement of the point to infinity, stability will be achieved. In other words, adding a tiny circumference and a point on that circumference which moves very quickly will not cause significant changes in the system in general.

Task #2: Modelling the Movement of Planets

The objective of this task is to *signify the convergence of trigonometric series through the stability of the planet’s trajectory, characterising it through the limit of the sequence of partial sums*. The expectation is that, as in Fourier’s work on heat propagation, the stable nature of the phenomenon leads to the need to talk about the convergence of trigonometric series (Farfán, 2012). Figure 5 shows the situation established.

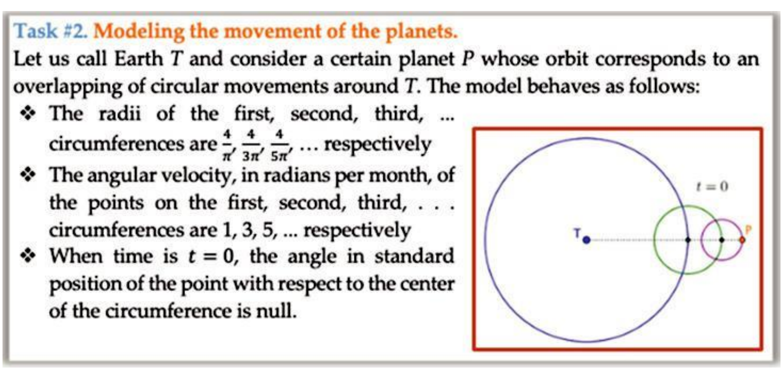
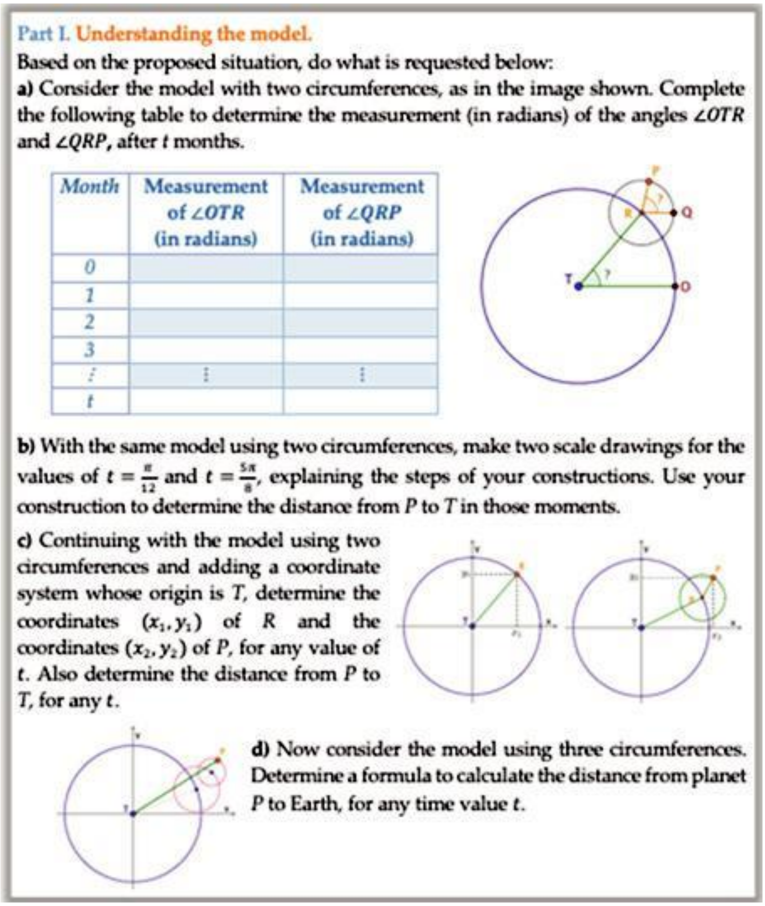


Figure 5. Situation raised in Task #2 of the learning situation.

This task is divided into two parts: the aim of the first part is for the student to understand how the system behaves, and the relationship it maintains with the data provided by the established situation. To begin with, item *a* (Figure 6) is intended to recognise the behaviour of the movement of the points on each circumference.



Part I. Understanding the model.

Based on the proposed situation, do what is requested below:

a) Consider the model with two circumferences, as in the image shown. Complete the following table to determine the measurement (in radians) of the angles $\angle OTR$ and $\angle QRP$, after t months.

Month	Measurement of $\angle OTR$ (in radians)	Measurement of $\angle QRP$ (in radians)
0		
1		
2		
3		
⋮	⋮	⋮
t		

b) With the same model using two circumferences, make two scale drawings for the values of $t = \frac{\pi}{12}$ and $t = \frac{5\pi}{8}$, explaining the steps of your constructions. Use your construction to determine the distance from P to T in those moments.

c) Continuing with the model using two circumferences and adding a coordinate system whose origin is T, determine the coordinates (x_1, y_1) of R and the coordinates (x_2, y_2) of P, for any value of t. Also determine the distance from P to T, for any t.

d) Now consider the model using three circumferences. Determine a formula to calculate the distance from planet P to Earth, for any time value t.

Figure 6. Task #2 – Part I, items *a* to *d* of the learning situation.

For this question, it is probable that the idea of angular velocity will generate difficulties, which was evidenced in the piloting and the different staging because the conception of radian itself causes difficulties, as reported in different investigations (Akkoc, 2008; Moore, 2009). Item *b* (Figure 6) is intended to verify the understanding of the behaviour of the system using a static system model (for a given time). The intention is that before looking for an algebraic procedure to determine the distance from *P* to *T*, the drawing is made to scale, and the distance determined on the same drawing.

Items *c* and *d* (Figure 6) seek to construct the first three partial sums of two trigonometric series, but with a meaning associated with the phenomenon – the coordinates of the planet – that allows us to determine the distance from the planet to the Earth. Students are expected to resort to drawing rectangular triangles on the figures provided to identify the trigonometric ratios involved, although generalisation to any time value requires that the trigonometric function be previously constructed (Montiel, 2011).

For the second part of this task, an applet (https://ggbm.at/jdBYwe8E) is provided in order to visualise the stability of the system. By manipulating the number of circumferences, it will be possible to analyse the behaviour of the system by studying how the planet's trajectory changes.

Part II. What if we increase the number of epicycles?
Based on what is observed in the provided applet, answer the following:
a) How does the trajectory of the planet change as more and more circumferences are added?
b) How does the distance from planet P to the Earth change as more and more circumferences are added for $t = \frac{\pi}{12}$, $t = \frac{3\pi}{4}$, $t = \pi$ and $t = 2\pi$? What could you say in general for each instant of time t?
c) Now consider the model using n circumferences. Determine a formula that allows you to calculate the distance from planet P to the Earth, for any time value t.
d) Based on your answers to questions (a) and (b), what is the behavior of the formula obtained in question (c)? Explain.

Figure 7. Task #2 – Part II, Items *a* to *d* of the learning situation.

Item *a* (Figure 7) expects the student to identify that the planet's trajectory stabilises as more circumferences are added. The answer is expected to emerge on its own based on the work carried out in Task #1.

Item *b* (Figure 7) is intended to provide a numerical analysis of the variation from one partial sum to another. The student is expected to realize that the trajectory of the planet is not stable for all time values and then to go on to mathematize the phenomenon in item *c* (Figure 7), as a generalization of the results obtained in Task #2 – Part I, in order to get that the distance from planet *P* to the Earth is equal to $\sqrt{x_n^2 + y_n^2}$ where:

$$\begin{cases} x_n = \sum_{k=1}^{n} \frac{4}{(2k-1)\pi} \cos[(2k-1)t] \\ y_n = \sum_{k=1}^{n} \frac{4}{(2k-1)\pi} \text{sen}[(2k-1)t] \end{cases}$$

Item *d* (Figure 7) leads to the interpretation of the mathematics involved in the physical phenomenon, relating the stability of the planet's trajectory to the limit of the partial sums involved in the formula calculated in item *c*. This is of utmost importance because in Fourier's work there was always a back-and-forth between the physical phenomenon and the mathematics involved, to validate the latter in the first (Farfán, 2012; Farfán & Romero, 2017).

The following questions for this task require another applet of GeoGebra (https:// ggbm.at/jf9gmZ4A), in which the model of the movement of planet *P* around Earth *T* is shown on the left and the change in the coordinates of planet P as time passes is on the right.

Part II. What if we increase the number of epicycles?

Based on what is observed in the provided applet, answer the following:

e) What is the behavior of the partial sums of the abscissas? And of the ordinates? Explain in your own words. What is the relationship between these answers and the answers to questions (a) and (b)?

f) Note that the obtained formulas correspond to partial sums of trigonometric series. Based on the same applet and your answer to the previous question, could you assure that the ordinate series converges or diverges? If it converges, could you identify its convergence value? Explain your answers.

g) If the range of values of t is changed for all those in which $t \geq 0$, what is the convergence value of the ordinate series of planet *P*?

h) Does it make sense in our model to consider negative values of t? What would the convergence value of the ordinate series of planet *P* be if we were to consider $t \in \mathbb{R}$?

Figure 8. Task #2 – Part II, items *e* to *h* of the learning situation.

Items *e* and *f* (Figure 8) expect that the students interpret the mathematics involved in the physical phenomenon, to relate the stability of the planet's trajectory with the convergence of the series involved in the formula calculated for the distance from planet *P* to Earth. They are expected to realise that the partial sum of the abscissas is not always divergent and that the partial sum of the ordinates is always convergent.

In particular, for subsection *f*, the hypothesis is that students will respond that the series of ordinates diverge when they in fact converge. We know this thanks to the pilot test, and the different staging since the main argument used was that "the graph is approaching two values 1 and -1", which provides evidence of the conception of functional limit as an obstacle to understanding the convergence of series. It is essential here to promote the confrontation of the arguments of the participants and, through shared dialogue, build a more robust response by questioning the nature of the terms of the series. In order to get, for example, that in the interval $(0,2\pi)$ the series of *P*'s ordinates converge to the function $f(t) = \begin{cases} 1 & if \quad 0 < t < \pi \\ -1 & if \quad \pi < t < 2\pi \end{cases}$.

While we know that in $t = \pi$, the series converges to zero – in fact, in all discontinuities considering $t \in \mathbb{R}$ – those students who glimpse convergence might think that near $t = \pi$ – the discontinuities – the series of *P*'s ordinates is divergent, since it has already been reported that students consider as convergent only that which converges uniformly and the rest is divergent (Albert, 1996). In addition, they would be observing the Gibbs phenomenon – many oscillations around the discontinuity – that will be the object of analysis in Task #4.

Items *g* and *h* (Figure 8) are intended to cause periodicity to be a result of the series and not a condition of the function being represented. The series, as a mathematical object, has specific characteristics that do not make sense in the phenomenon being modelled. In this case, this characteristic is the condition of periodicity, which occurs in the entire real straight line, but the phenomenon does not make sense when speaking of negative values of the time. To conclude with this task, it is expected that students will not be able to analytically write a periodic function in clause *h* since it is not usual to do so in school, they may write the function for an interval, and write with words that it repeats periodically.

Task #4: The Gibbs Phenomenon

This task aims to *differentiate the type of convergence between points close to discontinuities and those which are not, in order to build a better understanding of the convergence of the series, through the study of* the *behaviour of the partial sums*. As discussed above, students consider that the oscillations shown near the discontinuities occur because the series is divergent at those points (Gibbs Phenomena).

Then, we return to the series constructed in the previous tasks to analyse the behaviour of the partial sums in the points near the discontinuities, dividing the task into two parts, one for Task #2 series and the other for Task #3 series.

Part I. Let us get back to Task #2.

a) Consider the applet used in Task #2, which showed the partial sums of the ordinates of planet *P* and answer the following question: how does the sequence of partial sums behave in values very close to $t = 0$, $t = \pi$ and $t = 2\pi$?
How is this behavior observed in *P*'s trajectory?

b) Use the provided applet and choose a value of $t \in (0, \pi)$ "far" from the interval endpoints and answer the following question: from which partial sum does the approximation given by the sum differ from the value of the limit function by less than 0.1?

c) Use the provided applet and choose two values of $t \in (0, \pi)$ "close" to each of the interval endpoints and answer the following question: from which partial sum does the approximation given by the sum differ from the value of the limit function by less than 0.1?

d) What is the difference between the behavior of the partial sums for points "close" to discontinuities and those "far" from discontinuities?

e) Is the series convergent for values close to $t = 0$? And for values close to $t = \pi$? Explain your answers.

f) What is the exact value of the partial sums when $t = 0$ and $t = \pi$? What can you say about the convergence in these values? Explain your answers.

g) Based on the work done in this part, you may suggest another limit function of the ordinate series of planet *P*.

Figure 9. Task #4 – Part I, items *a* to *g* of the learning situation.

For item *a* (Figure 9), the applet provided in Task #2 (https://ggbm.at/jf9gmZ4A) is used, and the student is expected to respond that the sum diverges or that she/he is not entirely sure of the convergence. For questions *b* and *c*, another GeoGebra applet (https:// ggbm.at/tpvjSaFt) is used. With these questions, the student is expected to realise that, in order to achieve the desired approximation, the points "close" to the discontinuities require partial sums of a much greater order than the sum required by the points "far" from the discontinuities.

The aim of this analysis is for the student to make a difference between "fast" and "slow" convergence in items *d* and *e* (Figure 9) so that in item *f* (Figure 9), she/he is convinced that the series is convergent even in discontinuities. Thus, will be able to restate the limit function, in item *g* (Figure 9), to the function with $t \in [0,2\pi]$:

$$f(t) = \begin{cases} 1 & if \quad 0 < t < \pi \\ 0 & if \quad t = 0 \ \vee \ t = \pi \\ -1 & if \quad \pi < t < 2\pi \end{cases}$$

The second part of this task corresponds to the analysis of the series worked in Task #3. The questions and intentionality are the same; the only change is that the analysis is only carried out for $t = 0$ and close values.

Task #6: Calculation of Coefficients

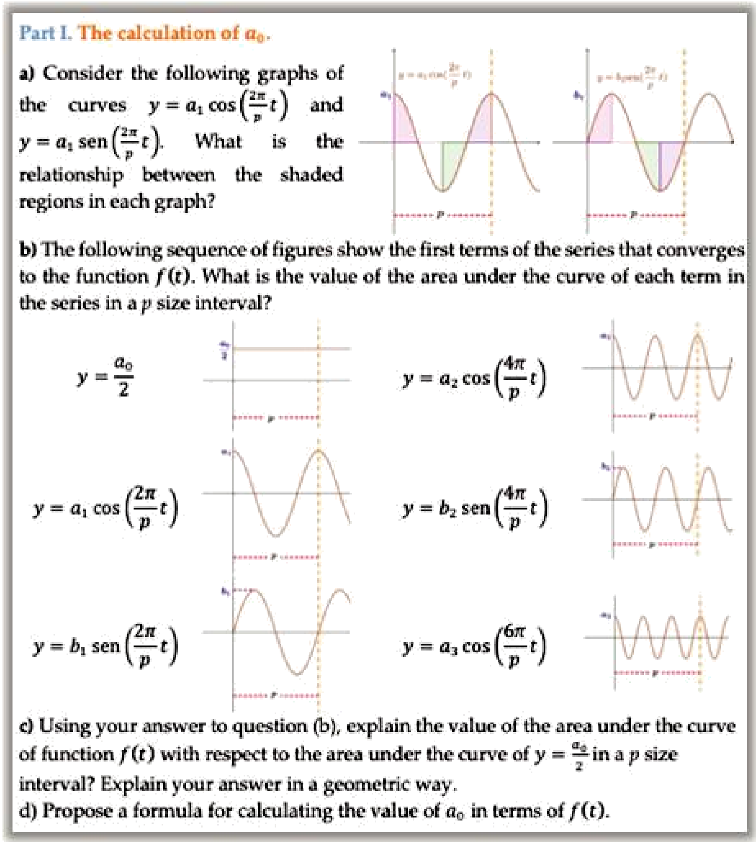
This task is intended to *signify the calculation of Fourier coefficients using graphical and geometric arguments, just like Fourier did*. It is essential, at the beginning of this task, to make the student aware that the reverse problem will be addressed, given the function to which a series converges, how to know the coefficients of that series. The situation is presented in Figure 10.

Task #6. Calculation of coefficients.
Now consider the reverse situation, i.e. that we know the function $f(t)$ to which converges the series whose partial sums are formed with the ordinate of planet P in a coordinate system. In other words, that the following holds:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{p}t\right) + b_k \, sen\left(\frac{2k\pi}{p}t\right)$$

Figure 10. Situation raised in Task #2 of the learning situation.

This task is divided into three parts, each one intended to signify the calculation of each of the Fourier coefficients a_0, a_k y b_k from the articulation of the geometric-analytical and algebraic registers, to validate the latter in the first.



Part I. The calculation of a_0.

a) Consider the following graphs of the curves $y = a_1 \cos\left(\frac{2\pi}{p}t\right)$ and $y = a_1 \text{sen}\left(\frac{2\pi}{p}t\right)$. What is the relationship between the shaded regions in each graph?

b) The following sequence of figures show the first terms of the series that converges to the function $f(t)$. What is the value of the area under the curve of each term in the series in a p size interval?

$y = \frac{a_0}{2}$ $y = a_2 \cos\left(\frac{4\pi}{p}t\right)$

$y = a_1 \cos\left(\frac{2\pi}{p}t\right)$ $y = b_2 \text{sen}\left(\frac{4\pi}{p}t\right)$

$y = b_1 \text{sen}\left(\frac{2\pi}{p}t\right)$ $y = a_3 \cos\left(\frac{6\pi}{p}t\right)$

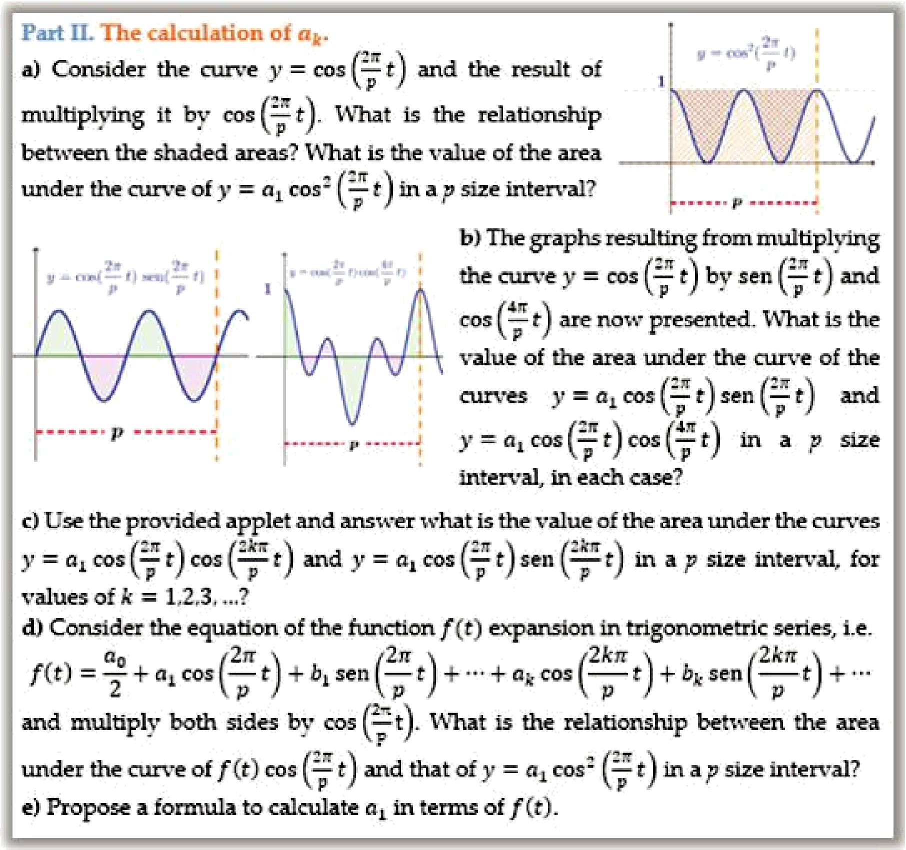
c) Using your answer to question (b), explain the value of the area under the curve of function $f(t)$ with respect to the area under the curve of $y = \frac{a_0}{2}$ in a p size interval? Explain your answer in a geometric way.

d) Propose a formula for calculating the value of a_0 in terms of $f(t)$.

Figure 11. Task #6 – Part I, items *a* and *b* of the learning situation.

The purpose of items *a* and *b* (Figure 11) is for the student to realise that the area under the curve of the series terms are all equal to zero, except that of the constant term $\frac{a_0}{2}$, which has an area of $\frac{a_0 p}{2}$ in a *p* size interval. Meanwhile, *c* and *d* (Figure 11) expect the student to use geometric arguments and their knowledge of defined integral to say that the area under the curve of the function $f(t)$ is equal to the sum of the areas under the curve of the series terms. Thus, the student can construct the formula for the calculation with a meaning associated with the notion of area the under the curve.

Regarding the calculation of the coefficient a_k, which corresponds to the second part of this task, it is intended to signify the calculation of the coefficients starting with coefficient a_1.



Part II. The calculation of a_k.

a) Consider the curve $y = \cos\left(\frac{2\pi}{p}t\right)$ and the result of multiplying it by $\cos\left(\frac{2\pi}{p}t\right)$. What is the relationship between the shaded areas? What is the value of the area under the curve of $y = a_1\cos^2\left(\frac{2\pi}{p}t\right)$ in a p size interval?

b) The graphs resulting from multiplying the curve $y = \cos\left(\frac{2\pi}{p}t\right)$ by $\text{sen}\left(\frac{2\pi}{p}t\right)$ and $\cos\left(\frac{4\pi}{p}t\right)$ are now presented. What is the value of the area under the curve of the curves $y = a_1\cos\left(\frac{2\pi}{p}t\right)\text{sen}\left(\frac{2\pi}{p}t\right)$ and $y = a_1\cos\left(\frac{2\pi}{p}t\right)\cos\left(\frac{4\pi}{p}t\right)$ in a p size interval, in each case?

c) Use the provided applet and answer what is the value of the area under the curves $y = a_1\cos\left(\frac{2\pi}{p}t\right)\cos\left(\frac{2k\pi}{p}t\right)$ and $y = a_1\cos\left(\frac{2\pi}{p}t\right)\text{sen}\left(\frac{2k\pi}{p}t\right)$ in a p size interval, for values of $k = 1,2,3,\ldots$?

d) Consider the equation of the function $f(t)$ expansion in trigonometric series, i.e.

$$f(t) = \frac{a_0}{2} + a_1\cos\left(\frac{2\pi}{p}t\right) + b_1\,\text{sen}\left(\frac{2\pi}{p}t\right) + \cdots + a_k\cos\left(\frac{2k\pi}{p}t\right) + b_k\,\text{sen}\left(\frac{2k\pi}{p}t\right) + \cdots$$

and multiply both sides by $\cos\left(\frac{2\pi}{p}t\right)$. What is the relationship between the area under the curve of $f(t)\cos\left(\frac{2\pi}{p}t\right)$ and that of $y = a_1\cos^2\left(\frac{2\pi}{p}t\right)$ in a p size interval?

e) Propose a formula to calculate a_1 in terms of $f(t)$.

Figure 12. Task #6 – Part II, items *a* to *e* of the learning situation.

In these questions (Figure 12) the student is expected to geometrically signify the orthogonality of the trigonometric functions and to use this fact to calculate the value of a_1. Thus, she/he will be able to identify with the help of an applet that the multiplication of $\cos\left(\frac{2\pi}{p}t\right)$ by any other term in the series – except the term $a_1\cos\left(\frac{2\pi}{p}t\right)$ – will have an area below the curve equal to zero (https://ggbm.at/nmkFwh2F), and on this basis, the value of $a_1\frac{p}{2}$ corresponds to the area under the curve resulting from multiplying the function $f(t)$ by $\cos\left(\frac{2\pi}{p}t\right)$. This permits the calculation of the value of a_1 in terms of $f(t)$, the known function.

It is then sought to generalise this reasoning to a_k using an applet similar to the one used to calculate a_1 (https://ggbm.at/bMz7yfhW). The questions in this section are intended to be the same as those proposed for the calculation of a_1. The idea is that they use similar arguments to conclude that the value of a_k is calculated using the formula

$\frac{2}{p}\int_0^p f(t)\cos\left(\frac{2k\pi}{p}t\right)dt$, the meaning of which is associated with the area under the curve. Finally, the third part of this task seeks to signify and construct the formula for the calculation of b_k so that the student uses strategies similar to those used to calculate a_k and thus conclude that $b_k = \frac{2}{p}\int_0^p f(t)\,\text{sen}\left(\frac{2k\pi}{p}t\right)dt$.

It is very likely that students will want to make explanations using definite integrals. However, it is essential that during sharing there is dialogue on the different possible arguments with the intention of comparing quality (the area) with formulae (the integral ones). Thus, the interaction between geometric-analytical and algebraic registers, needed to signify the calculation of Fourier coefficients, will be promoted (Romero, 2016; Farfán & Romero, 217).

FINAL THOUGHTS

The proposed learning situation seeks to contribute to the redesign of the SMd around the TFS since with a socio-epistemological foundation it intends to carry out a social construction of the series based on practices. Although it is true that there is still a need to analyse the interactions that would be provoked between the participants in some staging in order to compare the empirical with the theoretical results and validate the design by comparing the a posteriori analysis with the a priori analysis; the pilotages and the uncontrolled staging allow us to rescue some critical theoretical reflections.

According to Cantoral, Montiel and Reyes-Gasperini (2015), a proposal to redesign SMd should contain the following characteristics: functional character, diverse contextual rationalities, validation of knowledge and plurality of reference practices. In this respect, the proposed design presents the TFS as a prediction tool when analysing the model proposed by the Alexandrian astronomers for the planetary movement, where it is sought to model and interpret the system based on the superposition of circular movements, recognizing the *Prædiciere* as a social practice that rules the social construction of the TFS (functional character).

The choice of this physical-geometric environment responds to the reference practice of the target population, whose primary reference is school mathematics since their training does not encourage the transversality of mathematical notions. In this way, the evolution of trigonometric thinking is sought, where the situation posed causes the need to consider the convergence of the trigonometric series, a primordial characteristic that distinguishes it from the trigonometric function, whose arguments come from the context of the learner (contextual rationality and validation of knowledge).

Meanwhile, Montiel (2011) assures that, for the treatment of the trigonometric series, the trigonometric function must be deprived of all geometric origin, working exclusively with its analytical properties, particularly periodicity. From this research and in particular exemplified in the proposed learning situation, it is proposed that depriving the function of its geometric character is not necessary, instead, the design seeks to enhance this geometric character to give meaning to the trigonometric series, where the notion of convergence, addressed in the study of partial sums, becomes vital (plurality of reference practices for resignification).

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